Math 4220: Prelim 2 Practice Exam

This practice exam is much longer than the actual prelim!

1. (a) Let the contour Γ go from $z = 2$ to $z = -2$ counterclockwise around the upper semicircle $\{|z| = 2,\ \text{Im}(z) \geq 0\}$. By parametrizing Γ, compute

$$\int_\Gamma \frac{1}{z} dz.$$

(b) Let $L(z)$ be the branch of $\log(z)$ that takes the argument to be in the interval $(-\pi/2, 3\pi/2]$. Use $L(z)$ and the Fundamental Theorem of Calculus to compute the integral from part (a) by a different method.

2. Provide an example of an analytic function f with a pole of order 4 at $z_0 = 2i$ such that the residue $\text{Res}(f; 2i) = -3$.

3. Let f be analytic on the entire complex plane except at isolated singularities $z_1, \ldots, z_k$. Prove that f has an antiderivative if and only if $\text{Res}(f; z_j) = 0$ for every $1 \leq j \leq k$.

4. Let C be the unit circle oriented counterclockwise, and define

$$f(z) = \int_C \frac{\sin(w)}{w - z} dw, \qquad g(z) = \int_C \frac{\sin(w)}{(w - z)^2} dw$$

for z not on the circle. Compute: $f(\pi/6), f(\pi/3), g(\pi/6), g(\pi/3)$.

5. Recall the Cauchy estimates: If f is analytic on and inside the circle C_R of radius R centered at z_0, and $|f(z)| \leq M$ for all z on C_R, then for all $n \geq 0$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n}.$$

Suppose f is an entire function such that $|f(z)| \leq |z|^2$ for all $z \in \mathbf{C}$. Use the Cauchy estimates to prove that f must be a polynomial of degree at most 2, that is, $f(z) = c_0 + c_1 z + c_2 z^2$, and furthermore that $|c_2| \leq 1$. *Hint:* Consider the Taylor series for f centered at the origin.

6. Compute the Laurent series for $f(z) = \dfrac{2}{z - 1} + \dfrac{3}{z + 5}$ in the annulus $\{2 < |z| < 3\}$. What is the largest annulus $\{r < |z| < R\}$ on which the Laurent series converges?

7. Find the singularities of $f(z) = \dfrac{e^{1/z}[\cos(z) - 1]}{(z+\pi)^2(z+2\pi)^4}$ and classify them as removable, essential, or poles. Find the order of each pole.

8. What is the radius of convergence for the Taylor series of $f(z) = \dfrac{e^{\cos(z)}}{z^2+9}$ centered at -4?

9. Suppose that $g_1(z)$ and $g_2(z)$ are both analytic at z_0. Also assume that $g_1(z_0) \neq 0$, while g_2 has a simple zero at z_0, so the Taylor series are

$$g_1(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad \text{with } a_0 \neq 0,$$
$$g_2(z) = b_1(z - z_0) + b_2(z - z_0)^2 + \cdots \quad \text{with } b_1 \neq 0.$$

Prove that $f(z) = g_1(z)/g_2(z)$ has a simple pole at z_0, and that

$$\operatorname{Res}(f; z_0) = \frac{a_0}{b_1} = \frac{g_1(z_0)}{g_2'(z_0)}.$$

10. Use residue theory to compute $\displaystyle\int_0^{2\pi} \frac{1}{13 + 12\cos\theta}\,d\theta.$
Hint: $(2z+3)(3z+2) = 6z^2 + 13z + 6.$

11. Compute the residue of $f(z) = \dfrac{1}{z^2 + 2z^3}$ at $z = 0$.

12. Use residue theory to compute p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(2x)}{x+i}\,dx.$