Math 4220: Prelim 2

November 5, 2015

Name: Solutions

Instructions: This exam is 75 minutes long. It has 7 pages including the cover and 5 questions worth a total of 100 points. No written or electronic aids are allowed.

Please fully explain all your answers. If you need more space to answer a question, use the back of the *preceding* sheet or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

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1. (20 points) Suppose the functions f and g are both analytic on a domain that contains the closed unit disk $\{|z| \leq 1\}$. If $f(z) = g(z)$ for all z with $|z| = 1$, prove that $f(z) = g(z)$ for all z with $|z| < 1$.

Solution 1: By the Cauchy integral formula, for all z with $|z| < 1$,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw = \frac{1}{2\pi i}\int_C \frac{g(w)}{w - z}\,dw = g(z)$$

where C is the unit circle oriented counterclockwise, so that $f(w) = g(w)$ on C.

Solution 2: Let $F(z) = f(z) - g(z)$, so F is analytic on and inside the unit circle, and $F(z) = 0$ for all z on the unit circle. By the maximum modulus principle, $F(z)$ must be 0 for all z inside the unit circle, so $f(z) = g(z)$.

2. (20 points) Find the Laurent series for $f(z) = \dfrac{5}{z} + \dfrac{9}{z-4}$ centered at 1 that converges on the annulus $\{1 < |z-1| < 3\}$.

Hint: Write everything in terms of $w = z - 1$.

$$\frac{5}{z} + \frac{9}{z-4} = \frac{5}{w+1} + \frac{9}{w-3}.$$

If $1 < |w| < 3$,

then
$$\frac{5}{w+1} = \frac{5}{w} \cdot \frac{1}{1+\frac{1}{w}} = \frac{5}{w} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{w^n}$$

and
$$\frac{9}{w-3} = -\frac{9}{3} \cdot \frac{1}{1-\frac{w}{3}} = -3 \sum_{n=0}^{\infty} \frac{w^n}{3^n}.$$

Therefore,
$$\frac{5}{w+1} + \frac{9}{w-3} = \sum_{n=0}^{\infty} 5(-1)^n \cdot \frac{1}{w^{n+1}} - \sum_{n=0}^{\infty} \frac{w^n}{3^{n+1}}$$

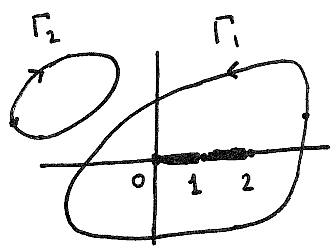
which means the Laurent series for $f(z)$ is

$$\sum_{n=1}^{\infty} 5(-1)^{n-1} \cdot \frac{1}{(z-1)^n} - \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} \cdot (z-1)^n.$$

3. (20 points) Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z-1} + \dfrac{c}{z-2}$, where $a, b, c \in \mathbf{C}$ are constants. Let $[0,2]$ denote the line segment $\{x + iy \in \mathbf{C} : 0 \le x \le 2,\ y = 0\}$, and let $D = \mathbf{C} \setminus [0,2]$. For which values of a, b, c does f have an antiderivative on D? Prove your answer.

We know f has an antiderivative on D if and only if $\int_\Gamma f(z)\,dz = 0$ for every simple closed contour Γ in D.

If Γ is such a contour, either it encloses none of the poles $0, 1, 2$ or it encloses all of them (see drawing).



(Γ_1 encloses all the poles, Γ_2 encloses none.)

Therefore, $\int_\Gamma f(z)\,dz = 0$ or

$\pm 2\pi i \left[\operatorname{Res}(0) + \operatorname{Res}(1) + \operatorname{Res}(2) \right]$

$= \pm 2\pi i \left[a + b + c \right]$, where the $\pm$ is because Γ may be positively or negatively oriented. (We know $\operatorname{Res}(0) = a$ because the function $\dfrac{b}{z-1} + \dfrac{c}{z-2}$ is analytic at 0, thus has a Taylor series, and so the Laurent series for $f(z)$ in a neighborhood of 0 is $\dfrac{a}{z}$ + (Taylor series). Alternatively,

(next page $\rightarrow$)

(This page for scratch work.)

$$\mathrm{Res}\,(0) = z f(z)\Big|_{z=0} = \left(a + \frac{bz}{z-1} + \frac{cz}{z-2}\right)\Big|_{z=0} = a.$$

Likewise, $\mathrm{Res}\,(1) = b$ and $\mathrm{Res}\,(2) = c.$)

In conclusion, the integral of $f(z)$ around every simple closed contour in D is zero if and only if $a+b+c=0$.

4. (20 points) Classify the singularity at 0 of $f(z) = \dfrac{z\sin(z)}{\sin(z) - z}$ as essential, removable, or a pole (whose order you must specify). Also find the residue of f at 0.

$$f(z) = \frac{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] - z} = \frac{z^2\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]}{z^3\left[-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots\right]}$$

$$= \frac{1}{z}\cdot g(z) \quad \text{where} \quad g(z) = \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots}.$$

We know g is analytic at 0 and $g(0) = \dfrac{1}{-1/3!} = -6$,

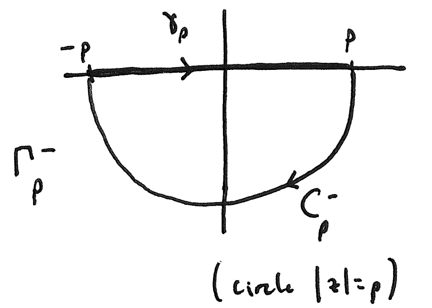
therefore we can write $g(z) = -6 + c_1 z + c_2 z^2 + \cdots$

and so $f(z) = \dfrac{-6}{z} + c_1 + c_2 z + \cdots$ has a pole of order 1 at 0 with residue -6.

5. (20 points) Using residue theory, compute p.v. $\int_{-\infty}^{\infty} e^{-ix} \frac{x}{x^2+4} dx$.

Define the contours γ_ρ, C_ρ^-, and $\Gamma_\rho^- = \gamma_\rho + C_\rho^-$

for fixed $\rho > 0$ as in the picture:



We know

$$\lim_{\rho\to\infty} \int_{\gamma_\rho} e^{-iz} \frac{z}{z^2+4} dz = \text{p.v.} \int_{-\infty}^{\infty} e^{-ix} \frac{x}{x^2+4} dx,$$

$$\lim_{\rho\to\infty} \int_{C_\rho^-} e^{-iz} \frac{z}{z^2+4} dz = 0$$ by Jordan's Lemma.

The function $e^{-iz} \frac{z}{z^2+4} = e^{-iz} \frac{z}{(z+2i)(z-2i)}$ has poles at $\pm 2i$, so

for all $\rho > 2$, $\int_{\Gamma_\rho^-} e^{-iz} \frac{z}{z^2+4} dz = -2\pi i \operatorname{Res}(-2i) = \left.\left(-2\pi i\, e^{-iz} \frac{z}{z-2i}\right)\right|_{z=-2i}$

(negatively oriented)

$= -2\pi i \cdot e^{-2} \left(\frac{-2i}{-4i}\right) = \frac{-\pi i}{e^2}$. Thus

$$\text{p.v.} \int_{-\infty}^{\infty} e^{-ix} \frac{x}{x^2+4} dx + 0 = \lim_{\rho\to\infty} \int_{(\gamma_\rho + C_\rho^-)} e^{-iz} \frac{z}{z^2+4} dz = \frac{-\pi i}{e^2}.$$