Math 4740   **Name:** _____

Spring 2015
Prelim
3/2/15
Time Limit: 50 Minutes

This exam contains 4 pages (including this cover page) and 3 problems. Check to see if any pages
are missing.

You may *not* use books, notes, calculator, phone, tablet, laptop or any other device on this exam.

- **Explain your answers**. A correct answer with no expla-
  nation may receive no credit.

- **Organize your work**. Equations scattered all over the
  page without a clear ordering may receive no credit.

- **If you use a theorem** from class or from the book, be
  sure to explain why the theorem may be applied.

- **If you need more space, use the back of the pages**
  and clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	10	
Total:	50	

1. (20 points) Let X_n be a Markov chain with states a, b, c, d, e and transition matrix

$$p = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{ccccc} a & b & c & d & e \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

(a) (3 points) List all of the irreducible closed sets.

(b) (2 points) Which states are recurrent?

(c) (5 points) Find $\lim_{n \to \infty} p^n(a, b)$.

(d) (5 points) Find $P_c(T_b < T_e)$.

(e) (5 points) Find $\lim_{n \to \infty} p^n(c, b)$.

2. (20 points) Let X_n be the last digit of $Y_1 + \ldots + Y_n$ where the Y_i are independent with

$$P(Y_1 = 1) = P(Y_i = 5) = P(Y_i = 9) = \frac{1}{3}.$$

(a) (2 points) X_n is random walk on a graph with 10 vertices. Draw the graph.

(b) (3 points) Find $P_0(X_3 = 3)$.

(c) (2 points) Is this chain irreducible? Is it aperiodic?

(d) (3 points) Find $\pi(0)$ and $E_0 T_0$.

(e) (5 points) Find $E_0 V$ where $V = \min\{n \geq 0 : 2 \leq X_n \leq 8\}$.

(f) (5 points) Find the value of the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k$$

and explain your answer.

3. (10 points) The Hydra starts with 3 heads. Each time Hercules swings his sword he cuts off all existing heads. When a head is cut it grows back into 3 new heads with probability $p = 4/7$, independently of the past and all other heads.

Let X_n be the number of heads after n sword swings. For example, $P(X_1 = 9) = (4/7)^3$.

(a) (3 points) Find $P(X_2 = 0)$. (You need not simplify your answer.)

(b) (2 points) Is X_n a Markov chain? What is the state space?

(c) (2 points) Is "3 heads" a positive recurrent, null recurrent or transient state?

(d) (3 points) What is the probability Hercules defeats the Hydra (i.e., there exists some n such that $X_n = 0$)?