Math 4740: Practice Final Exam Solutions

Spring 2016

1. (a) Write a transition matrix P for a Markov chain (X_n) on the state space $\{1, 2, 3, 4, 5\}$ such that:

- States $1, 2, 3$ are recurrent, while states $4, 5$ are transient.
- The unique stationary distribution is $\begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \end{bmatrix}$.
- The Markov chain does NOT converge to this stationary distribution as time tends to infinity.

There are many possible solutions. Here is one:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(b) Let $f(x) = 2x$. What is

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i)?$$

Let π be the stationary distribution given above. The desired limit is

$$\sum_{x=1}^{5} \pi(x) f(x) = (1/3)(2 + 4 + 6) = 4.$$

(c) Still using $f(x) = 2x$, modify your transition matrix P so that the conditions in part (a) remain satisfied but

$$\mathbf{E}_4 \left[\frac{1}{10} \sum_{i=1}^{10} f(X_i) \right] \geq 9.$$

In order for the average value of $f(X_i)$ for $1 \le i \le 10$ to be at least 9, it is required that $f(X_i) = 5$ with high probability. To meet the conditions in part (a), let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.01 & 0 & 0.99 \end{bmatrix}.$$

Starting from $X_0 = 4$, one has $X_1 = X_2 = \cdots = X_{10} = 5$ with probability $0.99^9 > 0.91$. Therefore,

$$\mathbf{E}_4 \left[\frac{1}{10} \sum_{i=1}^{10} f(X_i) \right] \ge \mathbf{P}_4(X_1 = \cdots = X_{10} = 5) \times 10 > 9.1.$$

2. Let $\{N(t)\}$ be a Poisson process with rate λ, and $X_n = N(n)$ for integers $n \ge 0$. Is (X_n) a Markov chain? If not, explain why not. If so, explain why it is, and write a formula for the transition probabilities $P(i, j)$.

The Markov property is that given the history $(X_0, \ldots, X_n)$, the value of X_{n+1} depends only on X_n and not the previous values of X_i for $i < n$. In this case, $X_{n+1} - X_n = N(n+1) - N(n)$ is independent of $\{N(t) : 0 \le t \le n\}$ by the independence of increments of the Poisson process. Therefore (X_n) is a Markov chain.

The transition probabilities are $P(i, j) = \mathbf{P}(N(n+1) = j \mid N(n) = i) = \mathbf{P}(N(n+1) - N(n) = j - i)$. Since $N(n+1) - N(n) \sim \text{Poisson}(\lambda)$,

$$P(i, j) = \mathbf{P}(\text{Poisson}(\lambda) = j - i) = e^{-\lambda} \cdot \frac{\lambda^{j-i}}{(j-i)!} \qquad \text{when } i \le j$$

and $P(i, j) = 0$ when $i > j$.

3. Let $p_0, p_1, \ldots$ be probabilities such that $\sum_{k=0}^{\infty} p_k = 1$. Consider the branching process (X_n) that evolves as follows: If $X_n = m$, each of the m individuals independently has a random number of children (having k children with probability p_k) and then dies, so that X_{n+1} is the total number of children produced by the individuals in the nth generation. Let $\mu = \sum_{k=0}^{\infty} k p_k < \infty$ be the average number of children per individual.

(a) Prove that X_n/μ^n is a martingale.

It suffices to check that

$$\mathbf{E}\left[\frac{X_{n+1}}{\mu^{n+1}} - \frac{X_n}{\mu^n} \;\middle|\; X_0 = x_0, \ldots, X_n = x_n\right] = 0,$$

or equivalently that

$$\mathbf{E}[X_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n] = \mu x_n. \tag{1}$$

(The other condition is that $\mathbf{E}[|X_n|] < \infty$. However, if we have verified (1), it follows that $\mathbf{E}[X_n] < \infty \implies \mathbf{E}[X_{n+1}] < \infty$, so by induction every $\mathbf{E}[X_n] < \infty$, and since $X_n \geq 0$ the absolute values are irrelevant.)

To check (1), assume there are x_n individuals at time n. Let Y_i be the number of children of the ith individual, so that the Y_i are iid with expectation μ. Then $X_{n+1} = Y_1 + Y_2 + \cdots + Y_{x_n}$ has expectation μx_n.

(b) Assume that $p_1 < 1$. Explain why for every $m > 0$,

$$\mathbf{P}(X_{n+j} = m \text{ for all } j \geq 0 \mid X_n = m) = 0.$$

Let $P(m, m) = \mathbf{P}(X_{i+1} = m \mid X_i = m)$, then the desired quantity is

$$\prod_{i=n}^{\infty} \mathbf{P}(X_{i+1} = m \mid X_i = m) = \prod_{i=n}^{\infty} P(m, m),$$

which is zero as long as $P(m, m) < 1$. Choose some $j \neq 1$ for which $p_j > 0$. With probability p_j^m, each of the m individuals at time i has exactly j children, meaning that

$$\mathbf{P}(X_{i+1} = jm \mid X_i = m) \geq p_j^m > 0$$

and therefore $P(m, m) < 1$.

(c) Suppose that $\mu = 1$ and $p_1 < 1$, and the process is started from $X_0 = k$. Use the martingale convergence theorem and part (b) to show that the extinction probability $\mathbf{P}_k(X_n = 0 \text{ for some } n) = 1$.

By part (a), (X_n) is a martingale. Because each $X_n \geq 0$, the martingale convergence theorem says that the sequence $(X_0, X_1, \ldots)$ converges to a limit with probability 1. The X_n are integer-valued, so convergence to a limit means that the sequence is eventually constant. Part (b) implies that the probability of the sequence being eventually constant at any value $m > 0$ is zero. Therefore with probability 1, the sequence is eventually constant at zero, which means that the population goes extinct.

4. Let $\{N(t)\}$ be a Poisson process with rate 4. Compute:

(a) $\mathbf{P}(N(2) = 1)$

$N(2) \sim \text{Poisson}(8)$ so $\mathbf{P}(N(2) = 1) = e^{-8}(8^1/1!) = 8e^{-8}$.

(b) $\mathbf{E}[N(5) \mid N(2) = 1]$

Given that $N(2) = 1$, $N(5) - N(2) \sim \text{Poisson}(12)$ and has expectation 12. Hence $\mathbf{E}[N(5) \mid N(2) = 1] = 1 + \mathbf{E}[N(5) - N(2)] = 13$.

(c) $\text{Var}(N(5) \mid N(2) = 1)$

$\text{Var}(N(5) \mid N(2) = 1) = \text{Var}(1 + N(5) - N(2)) = \text{Var}(N(5) - N(2)) = 12$.

(d) $\mathbf{E}[N(2) \mid N(5) = 20]$

Given that $N(5) = 20$, the 20 arrival times are distributed as independent uniform random variables on $[0, 5]$. Each one has probability $2/5$ of occurring at or before time 2, so the expected number of arrivals by time 2 is $(2/5) \times 20 = 8$.

5. Let $Y_1, Y_2, \ldots$ be iid normal random variables with mean 2 and variance 1. Let $\{N(t)\}$ be a Poisson process with rate λ, independent of the Y_i, and let $M(t) = Y_1 + \cdots + Y_{N(t)}$ (with $M(t) = 0$ if $N(t) = 0$). Prove that

$$\mathbf{E}\left[\frac{M(t)}{N(t)} \;\middle|\; N(t) > 0\right] = \frac{\mathbf{E}[M(t)]}{\mathbf{E}[N(t)]}.$$

The theorem on random sums from class gives $\mathbf{E}[M(t)] = \mathbf{E}[Y_i]\,\mathbf{E}[N(t)] = 2\,\mathbf{E}[N(t)]$, so the right side of the equation is 2. For the left side, for each

$k > 0$ we have

$$\mathbf{E}\left[\frac{M(t)}{N(t)} \,\middle|\, N(t) = k\right] = \frac{1}{k}\,\mathbf{E}[Y_1 + \cdots + Y_k] = \frac{2k}{k} = 2.$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left[\frac{M(t)}{N(t)} \,\middle|\, N(t) > 0\right] &= \frac{1}{\mathbf{P}(N(t) > 0)}\,\mathbf{E}\left[\frac{M(t)}{N(t)}; N(t) > 0\right] \\
&= \frac{1}{\mathbf{P}(N(t) > 0)} \sum_{k=1}^{\infty} \mathbf{P}(N(t) = k)\,\mathbf{E}\left[\frac{M(t)}{N(t)} \,\middle|\, N(t) = k\right] \\
&= \frac{1}{\mathbf{P}(N(t) > 0)} \sum_{k=1}^{\infty} \mathbf{P}(N(t) = k) \times 2 \\
&= 2.
\end{aligned}$$

6. The *Ehrenfest urn* process is a Markov chain (X_n) on $\{0, 1, \ldots, N\}$ whose transition probabilities are

$$P(i, i+1) = \frac{N-i}{N}, \qquad P(i, i-1) = \frac{i}{N}, \qquad P(i,j) = 0 \text{ otherwise.}$$

(a) Verify that $\pi(i) = \binom{N}{i}/2^N$ is the stationary distribution for this chain.

We must check that for each $0 \le j \le N$, $\sum_{i=0}^{N} \pi(i)P(i,j) = \pi(j)$. This means

$$\pi(j-1)P(j-1,j) + \pi(j+1)P(j+1,j) = \pi(j).$$

Say that $1 \le j \le N-1$. The left side is

$$\begin{aligned}
&\frac{1}{2^N}\left[\binom{N}{j-1} \cdot \frac{N-(j-1)}{N} + \binom{N}{j+1} \cdot \frac{j+1}{N}\right] \\
&= \frac{1}{2^N}\left[\frac{N!}{(j-1)!(N-j+1)!} \cdot \frac{N-j+1}{N} + \frac{N!}{(j+1)!(N-j-1)!} \cdot \frac{j+1}{N}\right] \\
&= \frac{1}{2^N}\left[\frac{(N-1)!}{(j-1)!(N-j-1)!} \cdot \left(\frac{1}{N-j} + \frac{1}{j}\right)\right] \\
&= \frac{1}{2^N}\left[\frac{(N-1)!}{(j-1)!(N-j-1)!} \cdot \frac{N}{j(N-j)}\right]
\end{aligned}$$

which equals the right side.

When $j = 0$ we must check that $\pi(1)P(1,0) = \pi(0)$, which (when multiplied by 2^N on both sides) is equivalent to

$$\binom{N}{1} \cdot \frac{1}{N} = \binom{N}{0}$$

which is true. Likewise, when $j = N$ we must check that $\pi(N-1)P(N-1, N) = \pi(N)$, which is equivalent to

$$\binom{N}{N-1} \cdot \frac{1}{N} = \binom{N}{N},$$

also true.

(b) Let $T_i = \min\{n \geq 0 : X_n = i\}$ and $h(i) = \mathbf{P}_i(T_0 < T_N)$. When $N = 4$, compute $h(i)$ for all $0 \leq i \leq 4$.

To start, $h(0) = 1$ and $h(4) = 0$. In between, we have the equations

$$\begin{aligned} h(1) &= \frac{1}{4}h(0) + \frac{3}{4}h(2) \\ h(2) &= \frac{1}{2}h(1) + \frac{1}{2}h(3) \\ h(3) &= \frac{3}{4}h(2) + \frac{1}{4}h(4). \end{aligned}$$

By symmetry around the midpoint of the interval, $h(2) = 1/2$. This means $h(1) = 1/4 + 3/8 = 5/8$ and $h(3) = 3/8 + 0 = 3/8$. (Even without noticing the symmetry, one could solve the system above without too much trouble.)

(c) Still with $N = 4$, let $T = \min\{n \geq 0 : X_n \in \{0, 4\}\}$. Show that $h(X_n)$ is not a martingale but $h(X_{T \wedge n})$ is a martingale.

The martingale property for $h(X_n)$ is that

$$\mathbf{E}[h(X_{n+1}) - h(X_n) \mid X_0 = x_0, \ldots, X_n = x_n] = 0.$$

Say that $X_n = 0$, then $X_{n+1} = 1$ with probability 1, so $h(X_{n+1}) < h(X_n)$ and the martingale property fails.

The martingale property for $h(X_{T\wedge n})$ is that

$$\mathbf{E}[h(X_{T\wedge(n+1)}) - h(X_{T\wedge n}) \mid X_0 = x_0, \ldots, X_n = x_n] = 0.$$

Suppose first that $x_0, \ldots, x_n$ are all between 1 and 3. Then $T > n$ and the expected value is

$$\mathbf{E}[h(X_{n+1}) - h(X_n) \mid X_0 = x_0, \ldots, X_n = x_n] = \mathbf{E}[h(X_{n+1}) \mid X_n = x_n] - h(x_n).$$

Since $1 \leq x_n \leq 3$, this quantity is zero because h satisfies the three equations we gave in the solution of part (b).

Now suppose that for some $i \leq n$ we have $x_i \in \{0, 4\}$. Then $T \leq n$ and the expected value is

$$\mathbf{E}[h(X_T) - h(X_T) \mid X_0 = x_0, \ldots, X_n = x_n] = 0.$$

7. Let $T_1, T_2, \ldots$ be the arrival times for a Poisson process with rate λ. For which real numbers r is $M_n = T_n - rn$ a supermartingale? A submartingale? A martingale?

We compute

$$\begin{aligned} &\mathbf{E}[M_{n+1} - M_n \mid T_1 = t_1, \ldots, T_n = t_n] \\ &= \mathbf{E}[T_{n+1} - r(n+1) - T_n + rn \mid T_1 = t_1, \ldots, T_n = t_n] \\ &= \mathbf{E}[T_{n+1} - T_n] - r \\ &= \frac{1}{\lambda} - r. \end{aligned}$$

Therefore (M_n) is a supermartingale when $r \geq 1/\lambda$, a submartingale when $r \leq 1/\lambda$, and a martingale when $r = 1/\lambda$.

(Note: $\mathbf{E}[|M_n|] \leq \mathbf{E}[T_n] + rn = n/\lambda + rn < \infty$.)

8. Let $\{N(t)\}$ be a Poisson process with rate λ and let $c > 0$. Prove that $\{N(ct)\}$ is also a Poisson process and find its rate.

It is enough to verify the three properties. First, $N(c \cdot 0) = 0$. Second, $\{N(ct)\}$ has independent increments: if $0 \leq s_1 \leq s_2 \leq \cdots \leq s_k$, then the increments $N(cs_{i+1}) - N(cs_i)$ are also disjoint increments of $\{N(t)\}$, so they are independent. Finally, if $0 \le s \le t$, $N(ct) - N(cs) \sim \text{Poisson}(\lambda(ct - cs))$. This means $\{N(ct)\}$ is a Poisson process with rate λc.

9. Suppose the price S_n of a stock at time n follows the binomial model with initial price $S_0 = 27$. At each time step the price is multiplied either by $u = 4/3$ or by $d = 2/3$. The interest rate is $r = 1/9$.

(a) Find the risk-neutral probability p^* that the stock goes up at any given time step.

By risk-neutrality, $p^*(S_0 u) + (1 - p^*)(S_0 d) = S_0(1 + r)$. Therefore, $p^*(4/3 - 2/3) + 2/3 = 10/9$ and so $p^* = (3/2)(4/9) = 2/3$.

(b) What is the current value of a European put option with strike price 30 and expiration time 2?

The possible values of S_2 with their risk-neutral probabilities are

$$\mathbf{P}^*(S_2 = 48) = \frac{4}{9}, \quad \mathbf{P}^*(S_2 = 24) = \frac{4}{9}, \quad \mathbf{P}^*(S_2 = 12) = \frac{1}{9}.$$

The expected payoff of the option is

$$\frac{4}{9} \times 0 + \frac{4}{9} \times 6 + \frac{1}{9} \times 18 = \frac{14}{3}$$

and its current value is $(81/100)(14/3) = \$3.78$.

(c) Without doing any additional computations, what can you say about the current value of an American put option with strike 30 and expiration time 2?

It is at least \$3.78. The American option gives more choices than the corresponding European option, so it must be worth at least as much.

(d) Repeat parts (b) and (c) for a call option with the same strike and expiration.

Let $V_P = \$3.78$ be the value of the European put and V_C be the value of the European call. By put-call parity,

$$V_P - V_C = \frac{30}{(1 + r)^2} - S_0 = 30 \times \frac{81}{100} - 27 = -\$2.70.$$

Therefore, $V_C = \$3.78 + \$2.70 = \$6.48$. This could also be computed directly using the method from (b).

The value of the American call is exactly \$6.48 since in our model it is always optimal to hold an American call until the expiration time.

(e) Suppose someone offers to sell you the put from part (b) for \$1 cheaper than the fair value in your answer. Describe how to implement an arbitrage strategy.

The idea is to buy the put for \$2.78 and also buy a portfolio that perfectly hedges the risk.

Suppose that the stock goes up from time 0 to 1, so $S_1 = 36$. The two possibilities for S_2 are 48 and 24, which correspond to option payoffs of 0 and 6. To *replicate* the option, we should hold $(0-6)/(48-24) = -1/4$ shares of stock between times 1 and 2. To *hedge* the option, which is what we want, we should hold $+1/4$ shares of stock over this interval.

Suppose that the stock goes down from time 0 to 1, so $S_1 = 18$. The two possibilities for S_2 are 24 and 12, which correspond to option payoffs of 6 and 18. To hedge the option, we should hold $(18-6)/(24-12) = 1$ share of stock over this interval.

Finally, we need the strategy at time 0. The two possibilities for S_1 are 36 and 18. If $S_1 = 36$ the expected value of the option in time 1 dollars is $(9/10)[(2/3) \cdot 0 + (1/3) \cdot 6] = \$1.80$. If $S_1 = 24$ the expected value of the option in time 1 dollars is $(9/10)[(2/3) \cdot 6 + (1/3) \cdot 18] = \$9$. Therefore to hedge the option, we should hold $(9-1.8)/(36-18) = 0.4$ shares of stock over this interval.

Here is the overall strategy. At time 0, buy the put for \$2.78 and buy 0.4 shares of stock for $0.4 \cdot 27 = \$10.80$. At time 1, if the stock has gone up to \$36, sell 0.15 shares of stock for \$5.40 and hold onto the remaining 0.25 shares. Then at time 2 there are two choices. If the final price is \$48, sell the 0.25 shares for \$12, choose not to exercise the put, and get a total profit in time 0 dollars of $(81/100) \cdot 12 - 2.78 - 10.80 + (9/10) \cdot 5.40 = \$1$. If the final price is \$24, sell the 0.25 shares for \$6 and exercise the put for another \$6. The total profit in time 0 dollars is also \$1.

If instead the stock goes down to \$18 at time 1, buy 0.6 more shares of stock for \$10.80 for a total of 1 share. At time 2 there are two choices. If the final price is \$24, sell the stock for \$24 and exercise the put for \$6 for a total profit in time 0 dollars of $(81/100) \cdot 30 - 2.78 - 10.80 - (9/10) \cdot 10.80 = \$1$. If the final price is \$12, sell the stock for \$12 and exercise the put for \$18 so that the total profit in time 0 dollars is still \$1.

10. A critic of the Black-Scholes model makes the following argument. "The model predicts that stock prices follow a log-normal distribution. The price of a stock at time t is predicted to be

$$S_t = S_0 e^{\mu t + \sigma \sqrt{t} Z} \tag{2}$$

where $Z \sim N(0,1)$ is a standard normal random variable, and $\mu = r - \sigma^2/2$ where r is the interest rate. I estimated μ and σ for a variety of stocks using historical data and found that the relationship $\mu = r - \sigma^2/2$ usually does not hold. Since the stock price does not evolve as predicted, the option prices given by the Black-Scholes model are wrong." Do you think this is a valid critique? What is your response?

The critic confuses the *actual* probability measure on stock prices with the *risk-neutral* measure. Imagine for example that the historical data was taken from a time period when stock prices tended to go up (i.e. a bull market). Looking back, investing in stocks would have led to greater profits than putting the money in the bank (or money market account) to get the risk-free interest rate. So, the critic would have found that $\mu > r - \sigma^2/2$, which is another way of saying that the stock prices (discounted by the interest rate) did not behave as martingales in real life.

This is quite possible, but it does not affect the validity of the Black-Scholes formula. The relationship $\mu = r - \sigma^2/2$ is supposed to hold under the risk-neutral measure, which is a fiction designed only to make it easier to compute option prices. Under the hypothetical bull market scenario, one could have made money by investing in the stock market; but given a mispriced option, there would have been an arbitrage opportunity as in problem 9(e).

The kernel of truth in the critic's argument is that Black-Scholes *does* assume that stock prices will follow a log-normal distribution, and to the extent this assumption is violated (which it is in practice to some degree), the formulas for option prices will be incorrect. This critique says that the equation (2) is mis-specified. Once the critic accepts equation (2) and starts estimating values for μ and σ, the rest of their argument does not hold up.