Math 4740: Practice Final Exam

Spring 2016

1. (a) Write a transition matrix P for a Markov chain (X_n) on the state space $\{1, 2, 3, 4, 5\}$ such that:

- States $1, 2, 3$ are recurrent, while states $4, 5$ are transient.
- The unique stationary distribution is $\begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 \end{bmatrix}$.
- The Markov chain does NOT converge to this stationary distribution as time tends to infinity.

(b) Let $f(x) = 2x$. What is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i)?$$

(c) Still using $f(x) = 2x$, modify your transition matrix P so that the conditions in part (a) remain satisfied but

$$\mathbf{E}_4 \left[\frac{1}{10} \sum_{i=1}^{10} f(X_i) \right] \geq 9.$$

2. Let $\{N(t)\}$ be a Poisson process with rate λ, and $X_n = N(n)$ for integers $n \geq 0$. Is (X_n) a Markov chain? If not, explain why not. If so, explain why it is, and write a formula for the transition probabilities $P(i, j)$.

3. Let $p_0, p_1, \ldots$ be probabilities such that $\sum_{k=0}^{\infty} p_k = 1$. Consider the branching process (X_n) that evolves as follows: If $X_n = m$, each of the m individuals independently has a random number of children (having k children with probability p_k) and then dies, so that X_{n+1} is the total number of children produced by the individuals in the nth generation. Let $\mu = \sum_{k=0}^{\infty} k p_k < \infty$ be the average number of children per individual.

(a) Prove that X_n/μ^n is a martingale.

(b) Assume that $p_1 < 1$. Explain why for every $m > 0$,

$$\mathbf{P}(X_{n+j} = m \text{ for all } j \geq 0 \mid X_n = m) = 0.$$

(c) Suppose that $\mu = 1$ and $p_1 < 1$, and the process is started from $X_0 = k$. Use the martingale convergence theorem and part (b) to show that the extinction probability $\mathbf{P}_k(X_n = 0 \text{ for some } n) = 1$.

4. Let $\{N(t)\}$ be a Poisson process with rate 4. Compute:

(a) $\mathbf{P}(N(2) = 1)$

(b) $\mathbf{E}[N(5) \mid N(2) = 1]$

(c) $\mathrm{Var}(N(5) \mid N(2) = 1)$

(d) $\mathbf{E}[N(2) \mid N(5) = 20]$

5. Let $Y_1, Y_2, \ldots$ be iid normal random variables with mean 2 and variance 1. Let $\{N(t)\}$ be a Poisson process with rate λ, independent of the Y_i, and let $M(t) = Y_1 + \cdots + Y_{N(t)}$ (with $M(t) = 0$ if $N(t) = 0$). Prove that

$$\mathbf{E}\left[\frac{M(t)}{N(t)} \,\middle|\, N(t) > 0\right] = \frac{\mathbf{E}[M(t)]}{\mathbf{E}[N(t)]}.$$

6. The *Ehrenfest urn* process is a Markov chain (X_n) on $\{0, 1, \ldots, N\}$ whose transition probabilities are

$$P(i, i+1) = \frac{N-i}{N}, \qquad P(i, i-1) = \frac{i}{N}, \qquad P(i, j) = 0 \text{ otherwise.}$$

(a) Verify that $\pi(i) = \binom{N}{i}/2^N$ is the stationary distribution for this chain.

(b) Let $T_i = \min\{n \geq 0 : X_n = i\}$ and $h(i) = \mathbf{P}_i(T_0 < T_N)$. When $N = 4$, compute $h(i)$ for all $0 \leq i \leq 4$.

(c) Still with $N = 4$, let $T = \min\{n \geq 0 : X_n \in \{0, 4\}\}$. Show that $h(X_n)$ is not a martingale but $h(X_{T \wedge n})$ is a martingale.

7. Let $T_1, T_2, \ldots$ be the arrival times for a Poisson process with rate λ. For which real numbers r is $M_n = T_n - rn$ a supermartingale? A submartingale? A martingale?

8. Let $\{N(t)\}$ be a Poisson process with rate λ and let $c > 0$. Prove that $\{N(ct)\}$ is also a Poisson process and find its rate.

9. Suppose the price S_n of a stock at time n follows the binomial model with initial price $S_0 = 27$. At each time step the price is multiplied either by $u = 4/3$ or by $d = 2/3$. The interest rate is $r = 1/9$.

(a) Find the risk-neutral probability p^* that the stock goes up at any given time step.

(b) What is the current value of a European put option with strike price 30 and expiration time 2?

(c) Without doing any additional computations, what can you say about the current value of an American put option with strike 30 and expiration time 2?

(d) Repeat parts (b) and (c) for a call option with the same strike and expiration.

(e) Suppose someone offers to sell you the put from part (b) for \$1 cheaper than the fair value in your answer. Describe how to implement an arbitrage strategy.

10. A critic of the Black-Scholes model makes the following argument. "The model predicts that stock prices follow a log-normal distribution. The price of a stock at time t is predicted to be

$$S_t = S_0 e^{\mu t + \sigma \sqrt{t} Z}$$

where $Z \sim N(0,1)$ is a standard normal random variable, and $\mu = r - \sigma^2/2$ where r is the interest rate. I estimated μ and σ for a variety of stocks using historical data and found that the relationship $\mu = r - \sigma^2/2$ usually does not hold. Since the stock price does not evolve as predicted, the option prices given by the Black-Scholes model are wrong." Do you think this is a valid critique? What is your response?