MATH 4740 Spring 2016

HW1 Solution.

Textbook Exercise:

1.1. No. Consider the following:

$$\left.\begin{array}{l} \mathbb{P}(X_{n+1}=0 \mid X_n=1, X_{n-1}=0) = 0 \\ \mathbb{P}(X_{n+1}=0 \mid X_n=1, X_{n-1}=2) = \frac{1}{2} \end{array}\right\} \Rightarrow$$

$$\mathbb{P}(X_{n+1}=0 \mid X_n=1, X_{n-1}=0) \neq \mathbb{P}(X_{n+1}=0 \mid X_n=1, X_{n-1}=2)$$

which violates Markov Property. $\square$.

1.2.

$$p = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/25 & 8/25 & 16/25 & 0 & 0 & 0 \\ 0 & 4/25 & 12/25 & 9/25 & 0 & 0 \\ 0 & 0 & 9/25 & 12/25 & 4/25 & 0 \\ 0 & 0 & 0 & 16/25 & 8/25 & 1/25 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\square$

1.5.

(a) $\mathbb{P}(X_4=4 \mid X_1=1) = p^3(1,4) = 0.4^3 = 0.064$.

(b). $\mathbb{P}(X_4=0 \mid X_1=1) = p^3(1,0) = 0.4 \cdot 0.6 \cdot 0.6 + 0.6 \cdot 1 \cdot 1 = 0.744$.

$\square$.

1.6.

(a) $P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}$

(b). $\mathbb{P}(X_2 = A \mid X_0 = A) = P^2(A, A) = 3/4$

$\mathbb{P}(X_2 = B \mid X_0 = A) = P^2(A, B) = 1/8$

$\mathbb{P}(X_2 = c \mid X_0 = A) = P^2(A, c) = 1/8$

$\mathbb{P}(X_3 = B \mid X_0 = A) = P^3(A, B) = 13/32$. $\square$.

1.7.

(a). $P = \begin{array}{c} \\ RR \\ RS \\ SR \\ SS \end{array} \begin{array}{c} \begin{array}{cccc} RR & RS & SR & SS \end{array} \\ \begin{pmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix} \end{array}$

(b). $P^2 = \begin{pmatrix} 0.36 & 0.24 & 0.24 & 0.16 \\ 0.36 & 0.24 & 0.12 & 0.28 \\ 0.36 & 0.24 & 0.24 & 0.16 \\ 0.18 & 0.12 & 0.21 & 0.49 \end{pmatrix}$

(c). Two cases satisfy :

	Sunday	Mon	Tues	Wed	
①	S	S	S	R	: $SS \to SS \to SR$
②	S	S	R	R	: $SS \to SR \to RR$

Thus $\mathbb{P}(\text{rainy Wed} \mid \text{Sunny Sunday \& Mon}) = P^2(SS, SR) + P^2(SS, RR)$
$= 0.39$. $\square$.

Additional Problems :

1. (a).

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0. & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b).

$$\lim_{n \to \infty} P^n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.58 & 0 & 0 & 0 & 0.42 \\ 0.31 & 0 & 0 & 0 & 0.69 \\ 0.12 & 0 & 0 & 0 & 0.88 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c). If the gambler starts with \$1, in the long run he ~~s~~ will end up with winning \$4 with probability 0.42.

□.

2. (a). (b).

$$\begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = \begin{array}{ccc} [\,0.41 & 0.4 & 0.19\,] \\ \text{Low} & \text{mid} & \text{upper} \end{array}.$$

□.

3. Base case:

$m=1 \quad \mathbb{P}(X_{n+1}=j \mid X_n=i) = P(i,j)$ by definition.

Induction Hypothesis:

Suppose $m=k \quad \mathbb{P}(X_{n+k}=j \mid X_n=i) = P^k(i,j)$ holds.

then

$$\mathbb{P}(X_{n+k+1}=j \mid X_n=i)$$

$$= \sum_u \mathbb{P}(X_{n+k+1}=j, X_{n+k}=u \mid X_n=i)$$

$$= \sum_u \mathbb{P}(X_{n+k+1}=j \mid X_{n+k}=u, X_n=i)\,\mathbb{P}(X_{n+k}=u \mid X_n=i) \quad (*)$$

$$= \sum_u \mathbb{P}(X_{n+k+1}=j \mid X_{n+k}=u)\,\mathbb{P}(X_{n+k}=u \mid X_n=i) \quad (☆)$$

$$\cancel{= \sum_u \mathbb{P}(X_{n+}}$$

$$= \sum_u P(u,j)\,P^k(i,u)$$

$$= P^{k+1}(i,j) \quad (\Delta)$$

$(*)$ follows from rules of conditional probability.

(☆) follows from Markov property.

(Δ) follows from matrix multiplication.