Math 4740: Homework 1

Due Friday, February 5 in class.

Do not show your work for the computational details of matrix multiplication! In fact, I encourage you to use a calculator or computer to multiply matrices. I recommend WolframAlpha especially if you do not have a graphing calculator or a computer algebra system such as MATLAB or Mathematica.

Textbook exercises (from section 1.12): 1.1, 1.2, 1.5, 1.6, 1.7. For exercise 1.7 you may find example 1.10 helpful.

Additional problems:

1. Consider the Gambler's Ruin Markov chain where the probability of winning \$1 is $a = 0.6$ and the probability of losing \$1 is $b = 0.4$. Assume that you keep playing either until you lose all your money or you get up to \$4.

(a) Write the transition matrix P for the Markov chain.

(b) Use a calculator or computer to find P^n for large values of n. Try enough values of n to formulate a conjecture for $A = \lim_{n\to\infty} P^n$. Write down this conjecture, rounding the nonzero entries of A to two decimal places.

(c) You should get that $A(1,4) = 0.42$. (Remember that the rows and columns of A are indexed by $\{0,1,2,3,4\}$ and not $\{1,2,3,4,5\}$.) How should the gambler at the casino interpret this number?

2. Recall the social mobility Markov chain, with transition matrix

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Suppose that in generation n, 30% of families are lower class, 60% of families are middle class, and 10% of families are upper class.

(a) What are the proportions of families in the different classes in generation $n+1$?

(b) See if you can express the computation in part (a) as a matrix-vector product. *Hint:* Consider the row vector $\begin{bmatrix}0.3 & 0.6 & 0.1\end{bmatrix}$.

3. Prove by induction on m that if (X_n) is a Markov chain with transition matrix P,

$$\mathbf{P}(X_{n+m} = j \mid X_n = i) = P^m(i, j)$$

for all integers $m \geq 1$. (Incidentally, the statement is also true for $m = 0$ if we interpret P^0 to be the identity matrix.)