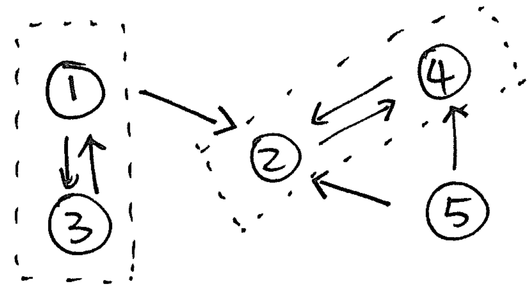
MATH 4740 Spring 2016

HW2 Solutions.

Textbook Exercise:

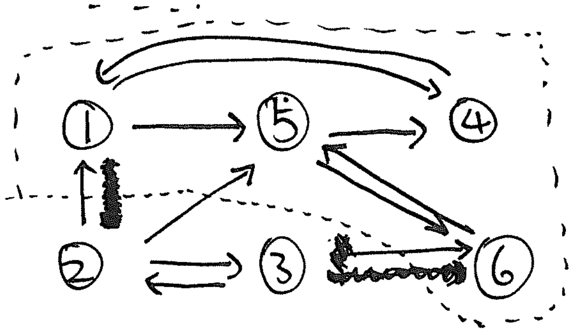
1.8. To identify recurrent / transient / irreducible states / sets of states, the most straightforward tool will be a diagram:

(a). 

Recurrent: $\{2,4\}$

Transient: $\{1,3,5\}$

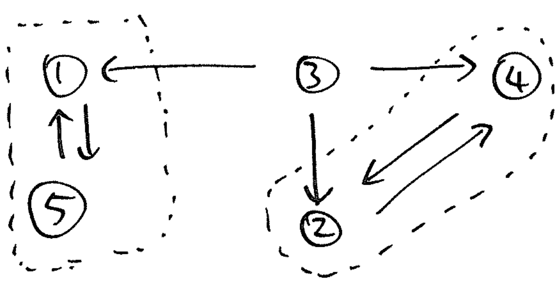
Closed Irreducible: $\{2,4\}$

(b). 

Recurrent: $\{1,4,5,6\}$

Transient: $\{2,3\}$

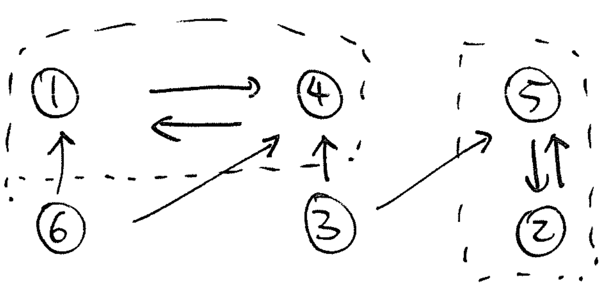
Closed irreducible: $\{1,4,5,6\}$.

(c). 

Recurrent: $\{2,4,1,5\}$

Transient: $\{3\}$

Closed irreducible: $\{2,4\}$ $\{1,5\}$

(d). 

Recurrent: $\{1,4,2,5\}$

Transient: $\{3,6\}$

Closed irreducible: $\{1,4\}$ $\{2,5\}$

$\square$

1.30. (a). $\left(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right)\begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.393 & 0.31 & 0.297 \\ \uparrow & \uparrow & \uparrow \\ L & C & G \end{pmatrix}$

(b) $\left(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right)\begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^7 = (0.3947\ \ 0.3070\ \ 0.2982)$

(c). To solve for stationary distribution π :

$$(\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} 0.5 & 0.2 & 0.3 & 1 \\ 0.4 & 0.5 & 0.1 & 1 \\ 0.25 & 0.25 & 0.5 & 1 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3\ 1)$$

$$\xRightarrow{\text{Row Reduction}} (\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} -0.5 & 0.2 & 1 \\ 0.4 & -0.5 & 1 \\ 0.25 & 0.25 & 1 \end{pmatrix} = (0\ 0\ 1)$$

$$\Rightarrow (\pi_1\ \pi_2\ \pi_3) = (0\ 0\ 1)\begin{pmatrix} -0.5 & 0.2 & 1 \\ 0.4 & -0.5 & 1 \\ 0.25 & 0.25 & 1 \end{pmatrix}^{-1}$$

$$= (0.3947\ \ 0.3070\ \ 0.2982).$$

$\square$

1.37. (a). The transition matrix is given by:

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{pmatrix}
\end{array}
$$

Reason:

(for example) If $X_n = 0$, the individual has no umbrella at the current location, then the other location must exist 3 umbrellas, and with probability 1 she will have access to 3 umbrellas at time $n+1$.

(b). $$(\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0.8 & 0.2 & 1 \\ 0 & 0.8 & 0.2 & 0 & 1 \\ 0.8 & 0.2 & 0 & 0 & 1 \end{pmatrix} = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ 1)$$

$$\Rightarrow \quad (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4) = \left(\frac{4}{19} \ \frac{5}{19} \ \frac{5}{19} \ \frac{5}{19}\right)$$

So $\mathbb{P}$(getting wet) $= \mathbb{P}$(she has no umbrella, and it's raining)

$$= \frac{4}{19} \cdot 0.2$$

$$= 0.042.$$

$\square$.

Additional Problems:

1. (a). $T = \min\{n \geq 1 : X_{n-1} = A,\ X_n = C\}$

(b). $T = T_A^2$

(c). No such stopping time exists.

Combining case 1 and 3, it violates the "deterministic" property of a stopping time.

2. Proof.

There are only two cases: either x communicates with y or not.

(i) Suppose x communicates with y.

Then by lemma 1.9, y is also recurrent.

Since x is recurrent, $\rho_{yx} = 1$ by lemma 1.6. It follows that since y is recurrent, $\rho_{xy} = 1$ by lemma 1.6 again.

(ii) Suppose x does not communicate with y.

then $\rho_{xy} = 0$ by definition.

$\square$.

3. (a). $E_x[N(x)] = \sum_{k=1}^{\infty} \mathbb{P}_x(N(x) \geq k)$ by (2)

$= \sum_{k=1}^{\infty} \rho_{xx}^k$ by (1)

$= \begin{cases} \dfrac{\rho_{xx}}{1-\rho_{xx}} & \text{if } \rho_{xx} < 1 \quad \text{(transient)} \\ \infty & \text{if } \rho_{xx} = 1 \quad \text{(recurrent)}. \end{cases}$

(b) By definition of indicator function, ~~$N(x)$~~ $\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=x\}}$ is the total counts when state x is visited, hence it represents the number of visits to x, i.e. $N(x)$.

(c). $\mathbb{E}_x[\mathbb{1}_{\{X_n=x\}}] = \mathbb{P}(X_n = x \mid X_0 = x) = p^n(x,x)$.

(d). <u>Proof.</u>

x is recurrent $\Leftrightarrow \rho_{xx} = 1 \overset{\text{part (a)}}{\Longleftrightarrow} E_x[N(x)] = \infty$

$\overset{\text{part (b)}}{\Longleftrightarrow} \mathbb{E}_x[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=x\}}] = \infty$

$\overset{\text{MCT}}{\Longleftrightarrow} \sum_{n=1}^{\infty} \mathbb{E}_x[\mathbb{1}_{\{X_n=x\}}] = \infty$ (Monotonic Convergence Thm)

$\overset{\text{part (c)}}{\Longleftrightarrow} \sum_{n=1}^{\infty} p^n(x,x) = \infty$

$\square$.