Math 4740: Homework 2

Due Friday, February 12 in class.

Textbook exercises (from section 1.12): 1.8, 1.30, 1.37. You may use any method you like to find stationary distributions, including any technique from section 1.4 or taking the limit of powers of the transition matrix.

Additional problems:

1. Suppose (X_n) is a Markov chain on the state space $\{A, B, C\}$.

Example: Find a stopping time T for (X_n) such that:

$$\text{If } (X_0, X_1, \ldots) = BAACAB\cdots, \text{ then } T = 3.$$
$$\text{If } (X_0, X_1, \ldots) = ACBBCC\cdots, \text{ then } T = 4.$$
$$\text{If } (X_0, X_1, \ldots) = AAACBA\cdots, \text{ then } T = 2.$$

A possible solution is $T = \min\{n \geq 2 : X_{n-2} = X_{n-1}\}$. (Remember that the sequences of letters start at time 0, not time 1.)

(a) Find a stopping time T for (X_n) such that:

$$\text{If } (X_0, X_1, \ldots) = ABAACB\cdots, \text{ then } T = 4.$$
$$\text{If } (X_0, X_1, \ldots) = CBBCAC\cdots, \text{ then } T = 5.$$
$$\text{If } (X_0, X_1, \ldots) = BACCBB\cdots, \text{ then } T = 2.$$

Or, prove that no such stopping time exists.

(b) Find a stopping time T for (X_n) such that:

$$\text{If } (X_0, X_1, \ldots) = AABACC\cdots, \text{ then } T = 1.$$
$$\text{If } (X_0, X_1, \ldots) = BCACCA\cdots, \text{ then } T = 5.$$
$$\text{If } (X_0, X_1, \ldots) = BABACB\cdots, \text{ then } T = 3.$$

Or, prove that no such stopping time exists.

(c) Find a stopping time T for (X_n) such that:

$$\begin{aligned}
&\text{If } (X_0, X_1, \ldots) = BACAAB\cdots, \text{ then } T = 1.\\
&\text{If } (X_0, X_1, \ldots) = ABABCC\cdots, \text{ then } T = 4.\\
&\text{If } (X_0, X_1, \ldots) = BAACAA\cdots, \text{ then } T = 2.
\end{aligned}$$

Or, prove that no such stopping time exists.

2. Let $\mathcal{X}$ be the finite state space of a Markov chain. Prove that if $x \in \mathcal{X}$ is recurrent, then for all $y \in \mathcal{X}$, either $\rho_{xy} = 0$ or $\rho_{xy} = 1$. You may use any result stated in class or the textbook.

3. Let (X_n) be a Markov chain. Recall that for any state x, $N(x) = \#\{n \geq 1 : X_n = x\}$. In class we saw that

$$\mathbf{P}_x(N(x) \geq k) = \rho_{xx}^k \quad \text{for all } k \geq 1. \tag{1}$$

This implies that $\mathbf{P}_x(N(x) = \infty) = 1$ if x is recurrent and 0 if x is transient.

The following formula is true for any random variable Y that takes nonnegative integer values:

$$\mathbf{E}[Y] = \sum_{k=1}^{\infty} \mathbf{P}(Y \geq k). \tag{2}$$

This can be proved by a quick but tricky argument: For each k, define the *indicator random variable* $\mathbf{1}_{\{Y \geq k\}}$ to equal 1 if $Y \geq k$ and 0 if not. Then (here's the trick) $Y = \sum_{k=1}^{\infty} \mathbf{1}_{\{Y \geq k\}}$, so

$$\mathbf{E}[Y] = \mathbf{E}\left[\sum_{k=1}^{\infty} \mathbf{1}_{\{Y \geq k\}}\right] = \sum_{k=1}^{\infty} \mathbf{E}\left[\mathbf{1}_{\{Y \geq k\}}\right] = \sum_{k=1}^{\infty} \mathbf{P}(Y \geq k).$$

(a) Use equations (1) and (2) to find a formula for $\mathbf{E}_x[N(x)]$ when x is transient. (Of course the expectation is ∞ when x is recurrent.)

(b) Explain in words why $N(x) = \sum_{n=1}^{\infty} \mathbf{1}_{\{X_n = x\}}$.

(c) Write a formula for $\mathbf{E}_x\left[\mathbf{1}_{\{X_n = x\}}\right]$ in terms of the transition matrix P.

(d) Prove that the state x is recurrent if and only if $\sum_{n=1}^{\infty} P^n(x, x) = \infty$.