MATH 4740 HW3 Solution

Textbook Exercise:

1.13 (a)

$$P^2 = \begin{pmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

(b). P is irreducible, so we have a unique π to it:

$$\pi \begin{pmatrix} -1 & 0 & 0.1 & 1 \\ 0 & -1 & 0.6 & 1 \\ 0.8 & 0.2 & -1 & 1 \\ 0.4 & 0.6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \pi = \left[\tfrac{4}{15} \quad \tfrac{7}{30} \quad \tfrac{1}{6} \quad \tfrac{2}{3} \right].$$

P^2 is not irreducible: $\{1,2\}$ $\{3,4\}$ are two communicating classes with a stationary distribution each:

$$P^2 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \Rightarrow \begin{matrix} \pi_1 P_1 = \pi_1 \\ \pi_2 P_2 = \pi_2 \end{matrix}$$

where $\pi_1 = \left[\frac{b}{a+b} \quad \frac{a}{a+b} \right] = \left[\frac{8}{15} \quad \frac{7}{15} \right]$

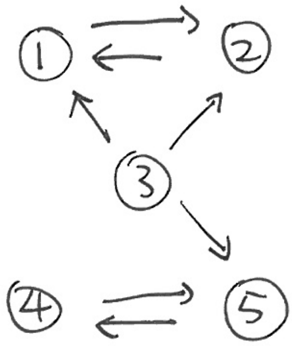
$$\pi_2 = \left[\tfrac{1}{3} \quad \tfrac{2}{3} \right]$$

Hence $\pi = \alpha \left[\frac{8}{15} \ \frac{7}{15} \ 0 \ 0 \right] + (1-\alpha) \left[0 \ 0 \ \frac{1}{3} \ \frac{2}{3} \right]$, $(\alpha \in [0,1])$

(c) $\lim_{n\to\infty} P^{2n}(x,x) = \begin{cases} \frac{8}{15} & x=1 \\ \frac{7}{15} & x=2 \\ 1/3 & x=3 \\ 2/3 & x=4 \end{cases}$ by Main Convergence Thm

Additional Problems:

1. (a)



two communicating classes:

$\{1,2\}$ $\{4,5\}$

are recurrent sets.

So let $P_1 = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix} \Rightarrow \pi_1 = (\frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0)$

$P_2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{pmatrix} \Rightarrow \pi_2 = (0\ 0\ 0\ \frac{1}{3}\ \frac{2}{3})$

(b). $\{\alpha\pi_1 + (1-\alpha)\pi_2\}$. $(\alpha \in [0,1])$.

(c). $\lim_{n\to\infty} \mu P^n = \left(\frac{a+b+0.4c}{2} \quad \frac{a+b+0.4c}{2} \quad 0 \quad \frac{0.6c+d+e}{3} \quad \frac{2(0.6c+d+e)}{3} \right.$

With p a+b → If starting from state 1 or 2, it will converge to π_1

With p e+d → If starting from state 4 or 5, it will converge to π_2.

With p c → If starting from state 3, it has $\frac{0.4}{0.3+0.1+0.6}$ chance that it will end up in $\{1,2\}$, and $\frac{0.6}{1}$ chace to $\{4,5\}$

(d) State 3 is still transient (0.1 will vanish).

So communicating classes of recurrent states ~~are~~ remain the same.

If starting from state 3, it will end up in $\{1, 2\}$ w.p. $\frac{0.3}{0.3+0.6} = \frac{1}{3}$ and in $\{4, 5\}$ w.p. $\frac{0.6}{0.3+0.6} = \frac{2}{3}$.

So now the ~~stationary di~~ limit is given by:

$$\lim_{n\to\infty} \mu P^n = \left(\frac{a+b+\frac{1}{3}c}{2} \quad \frac{a+b+\frac{1}{3}c}{2} \quad 0 \quad \frac{d+e+\frac{2}{3}c}{3} \quad \frac{2(d+e+\frac{2}{3}c)}{3} \right)$$

2. <u>Proof</u>.

Notice that $P^\infty P = P^\infty$. $\left(\lim_{n\to\infty} P^n P = \lim_{n\to\infty} P^{n+1} = P^\infty \right)$

Consider $P^\infty = \begin{pmatrix} - P_1 - \\ - P_2 - \\ \vdots \\ - P_n - \end{pmatrix}$ as a vector of rows,

then $P_i P = P_i \quad \forall i$ implying that $P_i = \pi \ \forall i$

by definition of stationary distribution.