MATH 4740 HW4 Solution.

Textbook Exercises:

1.14. A finite-state Markov Chain would converge if:

(i) it's irreducible

(ii) it's aperiodic.

(a). No, because each state has period 2.

(Also, notice it's an alternating transition matrix.)

(b). Yes. it's irreducible & aperiodic.

(c). No. Each state has period 3.

$\square$

1.36 The transition matrix:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0.05 & 0.95 & 0 \\ 0 & 0.02 & 0.98 \end{bmatrix}$$

(a) $$\pi = (0 \;\; 0 \;\; 1) \begin{pmatrix} -1 & 0 & 1 \\ 0.05 & -0.05 & 1 \\ 0 & 0.02 & 1 \end{pmatrix}^{-1} = \left(\frac{1}{71} \;\; \frac{20}{71} \;\; \frac{50}{71} \right).$$

Therefore the longrun fraction of time spent with 1 bulb is $\frac{20}{71}$.

(b). $E_0(T_0) = \frac{1}{\pi(0)} = 71.$ $\square$.

1.48 (a). The transition matrix is given by:

$$
P = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} &
\begin{bmatrix}
 & \frac{1}{2} & & & & & & & & & & \frac{1}{2} \\
\frac{1}{2} & & \frac{1}{2} & & & & & & & & & \\
 & \frac{1}{2} & & \frac{1}{2} & & & & & & & & \\
 & & \frac{1}{2} & & \frac{1}{2} & & & & & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & & & & \\
 & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
 & & & & & & \frac{1}{2} & & \frac{1}{2} & & & \\
 & & & & & & & \frac{1}{2} & & \frac{1}{2} & & \\
 & & & & & & & & \frac{1}{2} & & \frac{1}{2} & \\
 & & & & & & & & & \frac{1}{2} & & \frac{1}{2} \\
\frac{1}{2} & & & & & & & & & & \frac{1}{2} &
\end{bmatrix}
\end{array}
$$

First, notice that this is a ~~do~~ doubly stochastic chain, (the rows and columns of P both sum to 1).

By Thm 1.24, $\pi(x) = \frac{1}{N} \ \forall x \in \mathcal{X}$. ($N$ = # states).

So $E_x(T_x) = \frac{1}{\pi(x)} = \frac{1}{\frac{1}{N}} = N = 12$.

□.

Additional Problems:

1. (a). $\pi = \left(\frac{1}{3} \;\; \frac{2}{3}\right)$

(b). $\mathbb{P}_b(T_a = k) = \mathbb{P}(X_k = a, X_{k-1} = b, X_{k-2} = b, \cdots, X_1 = b \mid X_0 = b)$

$= \mathbb{P}(X_k = a \mid X_{k-1} = b)\, \mathbb{P}(X_{k-1} = b \mid X_{k-2} = b) \cdots$

$\cdot\, P(X_1 = b \mid X_0 = b)$

$= 0.3 \cdot 0.7^{k-1}$

$\mathbb{P}_b(T_a \geq k) = \mathbb{P}(X_{k-1} = b, X_{k-2} = b, \cdots, X_1 = b \mid X_0 = b)$

$= 0.7^{k-1}$

(c). $\mathbb{E}_b(T_a) = \sum_{k=1}^{\infty} \mathbb{P}_b(T_a \geq k)$

$= \sum_{k=1}^{\infty} 0.7^{k-1}$

$= \frac{1}{1-0.7}$

$= \frac{10}{3}$

(d). Starting from state a $(X_0 = a)$, it will stay at a w/ prob. 0.4 $(X_1 = a)$ or it will go to b w/ prob. 0.6. So $\mathbb{E}_a(T_a) = 0.4 \cdot 1 + 0.6 \cdot (\mathbb{E}_b(T_a) + 1)$

$= 3.$

$\pi(a) = \frac{1}{3} = \frac{1}{\mathbb{E}_a(T_a)}$. ✓

□

2. (a). Starting from state y, it has two possibilities:

it will never visit state x ($N(x) = 0$) or

it will visit x at some time point k.

In the latter case, once it's at state x, the number of visit to x after time k has the same distribution as $N(x)$ for the chain started at x, by strong Markov property.

So $\mathbb{E}_y[N(x)] = 0.(1-\rho_{yx}) + \rho_{yx} \cdot (1 + \mathbb{E}_x[N(x)])$.

(b) Suppose x is transient, then $\mathbb{E}_x[N(x)] < \infty$.

So $\mathbb{E}_y[N(x)] = \rho_{yx}(1 + \mathbb{E}_x[N(x)]) < \infty$

Also recall that $\mathbb{E}_y[N(x)] = \sum_{n=1}^{\infty} P^n(y,x) < \infty$.

So by test for divergence, $\lim_{n\to\infty} P^n(y,x) = 0$.

(c). Since $\lim_{n\to\infty} (\pi P^n)(x) = \pi(x)$, we have

$$\lim_{n\to\infty} \sum_{y\in\mathcal{X}} \pi(y) P^n(y,x) = \sum_y \lim_{n\to\infty} \pi(y) P^n(y,x)$$

$$= \sum_y \pi(y) \lim_{n\to\infty} P^n(y,x)$$

$$= \sum_y (\pi(y) \cdot 0)$$

$$= 0.$$

$$= \pi(x).$$

$\square$.

3. Notice that it's an irreducible Markov Chain with finite state space, we can apply the "main convergence theorem" such that $\pi(0) = \frac{1}{\mathbb{E}_0(T(0))}$

Starting from state 0, it will return to 0 in 3 steps ($0 \to -1 \to -2 \to 0$) or 4 steps ($0 \to 1 \to 2 \to 3 \to 0$), each w/ probability $\frac{1}{2}$. Hence

$$\mathbb{E}_0(T_0) = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 4 = \frac{7}{2}$$

Therefore $\pi(0) = \frac{1}{\frac{7}{2}} = \frac{2}{7}$.

$\square$.