# Math 4740: Homework 4

Due Friday, February 26 in class.

Textbook exercises (from section 1.12): 1.14, 1.36, 1.48(a).

For 1.36, to make the problem clearer, assume that bulbs always burn out during the daytime and are replaced overnight. Let $X_n$ be the number of working bulbs at the end of day $n$ and write the transition matrix for the Markov chain.

Additional problems:

1. Consider a Markov chain on the state space $\{a, b\}$ with transition matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}$$

where the first row/column correspond to state $a$ and the second row/column correspond to state $b$. Let $T_a = \min\{n \geq 1 : X_n = a\}$ be the first positive time that the chain visits state $a$.

(a) Find the stationary distribution $\pi$ of the Markov chain.

(b) Given an integer $k \geq 1$, write a formula for $\mathbf{P}_b(T_a = k)$. Also write a formula for $\mathbf{P}_b(T_a \geq k)$.

(c) Compute $\mathbf{E}_b[T_a]$. You may find useful the statement from HW #2 that if the random variable $Y$ takes only nonnegative integer values, then $\mathbf{E}[Y] = \sum_{k=1}^{\infty} \mathbf{P}(Y \geq k)$.

(d) Write a formula for $\mathbf{E}_a[T_a]$ in terms of $\mathbf{E}_b[T_a]$, and use your answer from part (c) to compute $\mathbf{E}_a[T_a]$. Verify that $\pi(a) = 1/\,\mathbf{E}_a[T_a]$.

2. For a Markov chain $(X_n)$ with transition matrix $P$, let $N(x) = \#\{n \geq 1 : X_n = x\}$ be the number of visits to state $x$ at positive times. In HW #2 you showed that

$$\mathbf{E}_x[N(x)] = \sum_{n=1}^{\infty} P^n(x, x),$$

and this sum is finite if and only if the state $x$ is transient. In that case, the "test for divergence" from calculus implies that $\lim_{n\to\infty} P^n(x,x) = 0$.

(a) For any other state $y$, let $\rho_{yx}$ be the probability that the Markov chain started at $y$ ever visits $x$. Prove that $\mathbf{E}_y[N(x)] = \rho_{yx}(1 + \mathbf{E}_x[N(x)])$.

(b) Deduce from part (a) and the test for divergence that if $x$ is transient, then $\lim_{n\to\infty} P^n(y,x) = 0$ for all states $y$.

(c) Prove that if $x$ is transient and $\pi$ is any stationary distribution for the Markov chain, then $\pi(x) = 0$. *Hint:* Use that

$$(\pi P^n)(x) = \sum_{\text{all states } y} \pi(y) P^n(y,x).$$

3. Consider the Markov chain from Example 4.4 (p.28 of the printed textbook, pp.23-24 of the online PDF).[1] Use the distribution of the first return time to state $\mathbf{0}$ to compute the value $\pi(\mathbf{0})$ of the stationary distribution.

[1] I have no idea why Example 4.4 is in the middle of Chapter 1 between Examples 1.22 and 1.23. There's also an Example 4.4 in Chapter 4.