MATH 4740 HW6 Solution

Textbook Exercises:

1.48 b) Denote $V_x = \min\{n \geq 0 : X_n = x\}$.

WLOG, consider that we start from "1", then the probability of interest becomes ~~$\mathbb{P}(V$~~

$$\mathbb{P}(V_{12} < V_1 \mid X_1 = 2)\,\mathbb{P}(X_1 = 2) + \mathbb{P}(V_2 < V_1 \mid X_2 = 12)\,\mathbb{P}(X_2 = 12).$$

Notice that if we suppose $X_1 = 2$, then $\mathbb{P}(V_{12} < V_1) = \frac{1}{N-1} = \frac{1}{11}$, by the formula found in Gambler's ruin problem where $p = \frac{1}{2}$. By symmetry, $\mathbb{P}(V_2 < V_1 \mid X_2 = 12) = \frac{1}{11}$.

Hence the probability of visiting all states before returning to the starting state is $\frac{1}{11} \cdot \frac{1}{2} + \frac{1}{11} \cdot \frac{1}{2} = \frac{1}{11}$.

1.74. (a). If 0 is recurrent then $\mathbb{P}(T_0 < \infty) = 1$.

$$\mathbb{P}(T_0 < \infty) = 1 - \mathbb{P}(T_0 = \infty)$$
$$= 1 - \prod_{i=0}^{\infty} p_i$$

So $\prod_{i=0}^{\infty} p_i$ will make 0 recurrent.

(b).(c). To make "0" positive recurrent, we need $E_0[T_0] < \infty$. So we find the stationary distribution first:

Denote the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$

Since the transition matrix $P = \begin{pmatrix} 1-P_0 & P_0 & 0 & 0 & 0 & \cdots \\ 1-P_1 & 0 & P_1 & 0 & 0 & \cdots \\ 1-P_2 & 0 & 0 & P_2 & 0 & \cdots \\ 1-P_3 & 0 & 0 & 0 & P_3 & \\ \vdots & & & & & \ddots \end{pmatrix}$

we have that $\pi P = \pi \Rightarrow$

$$\pi_0 = \pi_0$$
$$\pi_1 = \pi_0 P_0$$
$$\pi_2 = \pi_1 P_1 = P_1 P_0 \pi_0$$
$$\pi_3 = \pi_2 P_2 = P_2 P_1 P_0 \pi_0$$
$$\vdots$$
$$\pi_n = \left(\prod_{k=0}^{n-1} P_k\right) \pi_0 .$$

Also $\sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} P_k\right) \pi_0 + \pi_0 = 1 \Rightarrow \pi_0 = \dfrac{1}{1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} P_k}$

Therefore, to make "0" positive recurrent, we need

$$1 + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} P_k\right) < \infty \quad , \text{ i.e. } \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} P_k\right) < \infty .$$

The stationary distribution is given by

$$\begin{cases} \pi_0 = \dfrac{1}{\left(1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} P_k\right)} \\ \\ \pi_n = \dfrac{\left(\prod_{k=0}^{n-1} P_k\right)}{1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} P_k} \quad \text{for } n \neq 0 . \end{cases}$$

1.77 $\mu =$ "Expected number of offspring of one individual"

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_n \geq k)$$

$$= \sum_{k=1}^{\infty} (1-p)^k$$

$$= \frac{1-p}{p}$$

Consider 2 cases:

① $\mu \leq 1 \Rightarrow p \geq \frac{1}{2} \Rightarrow \rho = 1$. Extinction occurs with probability 1.

②. $\mu > 1 \Rightarrow p < \frac{1}{2}$.

$$\rho = \underset{\rho}{\operatorname{argmin}} \{ \phi(x) = x \}$$

where $\phi(x) = \sum_{k=0}^{\infty} p(1-p)^k x^k = p \sum_{k=0}^{\infty} (x(1-p))^k = \dfrac{p}{1-x(1-p)}$

$$\Rightarrow \rho = \frac{p}{1-\rho(1-p)} \qquad \Rightarrow \rho = \min\left(\frac{1 \pm \sqrt{1-4p(1-p)}}{2(1-p)} \right)$$

$$= \min\left(\frac{1 \pm |2p-1|}{2(1-p)} \right)$$

$$= \frac{p}{1-p}.$$

Above all, $\rho = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ \frac{p}{1-p} & \text{if } p < \frac{1}{2}. \end{cases}$