Math 4740: Homework 6

Due Friday, March 18 in class.

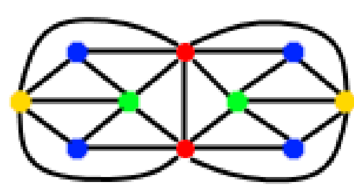
Textbook exercises (from section 1.12): 1.48(b), 1.74, 1.77.

Hints: For 1.48(b), a formula that was derived in class will be useful. For 1.74, it is probably easier to do part (c) before part (b).

Additional problem:

1. Recall that a graph is a set of vertices, some of which are connected by edges. Suppose we have a graph with finite vertex set V. To each vertex we assign a color from a finite list C of colors. This defines a function $f : V \to C$, which is called a *coloring* of the graph. The coloring is *proper* if for any vertices v, w that are connected by an edge, $f(v) \neq f(w)$. The famous Four-Color Theorem states that if the graph can be drawn on the plane with no edges crossing, and $C = \{c_1, c_2, c_3, c_4\}$ has four colors, then there exists a proper coloring $f : V \to C$. We consider the general case: any finite graph, any number of colors.

For any coloring $f : V \to C$, let $N(f)$ be the number of pairs of vertices v, w that are connected by an edge and that satisfy $f(v) = f(w)$. The proper colorings are precisely those for which $N(f) = 0$, and the larger the value of $N(f)$, the further the coloring is from being proper. The figure below shows a coloring for which $N(f) = 1$. (Note: the Four-Color Theorem guarantees that this graph has a proper coloring. Indeed, we could recolor the bottom three vertices as red, blue, red rather than blue, red, blue.)



Fix $0 < \alpha < 1$ and define a probability distribution on the space of all possible colorings $f : V \to C$ by

$$\pi(f) = \frac{\alpha^{N(f)}}{Z}, \quad \text{where } Z = \sum_{g:V\to C} \alpha^{N(g)}$$

is the appropriate normalizing constant. Since $\alpha < 1$, this distribution assigns greater weight to colorings that are closer to being proper.

(a) Consider the following Markov chain on the space of all colorings. Given a particular coloring, to take one step forward in the chain: choose a vertex uniformly at random, forget its current color, and recolor it with a new color chosen uniformly from C (which could be the original color with probability $1/|C|$). Leave all the other vertices alone. To take the next step in the chain, choose another vertex uniformly at random (which could be the same as the first vertex with probability $1/|V|$) and follow the same procedure.

Use the Metropolis algorithm, where the proposal chain is the one given above, to construct a new chain that has stationary distribution π. Describe the transition rule in words, precisely enough that someone could use your description to write a computer program that runs the Markov chain.

(b) Verify that your Metropolis chain is the same as the proposal chain when $\alpha = 1$.

(c) What is the transition rule for the Metropolis chain in the limit as $\alpha \to 0$?

(d) Explain why the example on the previous page is an absorbing state for the Metropolis chain with $\alpha = 0$, even though it is not a proper coloring.

(e) Do you think the Metropolis chain would be an efficient way of finding a proper coloring for a graph? What are the benefits and drawbacks to picking higher or lower values of α? Would the chain perform better or worse if more colors were available?