MATH 4740 HW7 Solution

2.2 $T \sim \exp(5)$

$$\mathbb{P}(T \geq 10 \mid T \geq 7) = \mathbb{P}(T \geq 10-7)$$
$$= e^{-\frac{1}{5}\cdot 3}$$
$$= e^{-\frac{3}{5}}$$

$\square$

2.4. Let T_i denote the lifetime of machine i.

Then $T_1 \sim \exp(\lambda_1)$ $T_2 \sim \exp(\lambda_2)$

$\mathbb{P}$(machine 2 fails prior to machine 1)

$$= \mathbb{P}(T_1 \geq t,\ T_2 < T_1 - t)$$
$$= \mathbb{P}(T_1 \geq t)\,\mathbb{P}(T_2 < T_1 - t \mid T_1 \geq t)$$
$$= \mathbb{P}(T_1 \geq t)\,\mathbb{P}(T_1 > T_2) \quad \text{("memoryless")}$$
$$= e^{-\lambda_1 t} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2}$$

$\square$.

2.6 Let $T_1 = \{\text{time to finish a manicure}\}$

$T_2 = \{\text{time to finish a haircut}\}$

$T_1 \sim \exp(\frac{1}{20})$ $T_2 \sim \exp(\frac{1}{30})$

(a). $\mathbb{P}(T_1 < T_2) = \dfrac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{30}} = \dfrac{3}{5}$

(b). $\mathbb{E}(\max\{T_1, T_2\}) = \dfrac{1}{\frac{1}{20}} + \dfrac{1}{\frac{1}{30}} - \dfrac{1}{\frac{1}{20}+\frac{1}{30}} = 38$ mins.

$\square$

2.9. Let $T_i = \{\text{time spent at server } i\}$

So $T_1 \sim \exp(\frac{1}{6})$ $\quad T_2 \sim \exp(\frac{1}{3})$

(a). $\mathbb{E}(\text{time for Bob to get goods and pay when Al is @ Server } 1\}$

$= \mathbb{E}(T_1) + \mathbb{E}(\max\{\text{time Bob spent at } 1, \text{ time Al spent at } 2\})$

$+ \mathbb{E}(T_2)$

$$= 6 + \left(6 + 3 - \frac{1}{\frac{1}{6}+\frac{1}{3}}\right) + 3$$

$= 16$ mins

(b). $$\frac{1}{\lambda} + \left(\frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\mu+\lambda}\right) + \frac{1}{\mu}$$

2.23. $N(t) \sim \text{Poisson}(3t)$

(a) $\mathbb{P}(N(2)=0) = e^{-3\cdot 2}\frac{(3\cdot 2)^0}{0!} = e^{-6}$.

(b) $T_1 \sim \exp(3)$.

Additional problems:

1. (a) $$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X>x)\,dx$$

$$= \sum_{k=0}^{\infty} \int_k^{k+1} \mathbb{P}(X>x)\,dx$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X>k) \int_k^{k+1} 1\,dx$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(X \geq k+1)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

(b) $\mathbb{E}(X) = \int_0^\infty t f_X(t)\,dt$

$= \int_0^\infty \int_0^t 1 \cdot f_X(t)\,dx\,dt$

$= \int_0^\infty \int_0^\infty \mathbb{1}_{\{x \leq t\}} f_X(t)\,dx\,dt$

$= \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx$

$= \int_0^\infty \mathbb{P}(X > x)\,dx$

$\square$

2 (a) $\mathbb{P}(\max\{S,T\} \leq t) = \mathbb{P}(S < t)\mathbb{P}(T \leq t)$

$= (1 - e^{-\lambda t})(1 - e^{-\mu t})$

(b) $\mathbb{E}(\max\{S,T\}) = \int_0^\infty \mathbb{P}(\max\{S,T\} > t)\,dt$

$= \int_0^\infty (1 - (1-e^{-\lambda t})(1-e^{-\mu t}))\,dt$

$= \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\lambda+\mu}$

(c) $\mathbb{E}(\min\{S,T\} + \max\{S,T\}) = \mathbb{E}(\min\{S,T\}) + \mathbb{E}(\max\{S,T\})$

$= \frac{1}{\lambda+\mu} + \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\lambda+\mu}$

$= \frac{1}{\lambda} + \frac{1}{\mu}$

$= \mathbb{E}(S+T)$

LHS = RHS.

$\square$

3. (a) $\mathbb{P}(X+Y \leq t \mid X=s) = \mathbb{P}(Y \leq t-s)$

$$= F_Y(t-s)$$

so $F_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(s) F_Y(t-s)\,ds.$

(b). $\int_{-\infty}^{t} g(r)\,dr = \int_{-\infty}^{t}\int_{-\infty}^{\infty} f_X(s) f_Y(r-s)\,ds\,dr$

$$= \int_{-\infty}^{\infty} f_X(s)\left(\int_{-\infty}^{t} f_Y(r-s)\,dr\right)ds$$

$$= \int_{-\infty}^{\infty} f_X(s) F_Y(t-s)\,ds$$

$$= F_{X+Y}(t) \qquad \text{from (a).}$$

By the fundamental theorem of calculus, we know that $\frac{d}{dt}\int_{-\infty}^{t} g(r)\,dr = g(t) = \frac{d}{dt}F_{X+Y}(t) = f_{X+Y}(t)$

$\square$.