Math 4740: Homework 7

Due Friday, March 25 in class.

Textbook exercises (from section 2.6): 2.2, 2.4, 2.6, 2.9, 2.23. Important note: Solve additional problem #2 first! You'll need it for these exercises.

Additional problems:

1. Recall from earlier problem sets that if X is a random variable taking nonnegative integer values, then

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k). \tag{1}$$

In fact, this is a special case of the following formula. If X is a real-valued random variable with $\mathbf{P}(X \geq 0) = 1$, then

$$\mathbf{E}[X] = \int_0^{\infty} \mathbf{P}(X > x)\, dx. \tag{2}$$

(a) Prove that formula (2) reduces to formula (1) if in fact X takes only nonnegative integer values.

(b) Suppose now that X is a continuous random variable with density function f_X, so that

$$\mathbf{E}[X] = \int_0^{\infty} t f_X(t)\, dt.$$

Prove (2) by writing $t = \displaystyle\int_0^t 1\, dx$ and switching the order of integration.

2. In class it was shown that if $S \sim \text{Exp}(\lambda)$ and $T \sim \text{Exp}(\mu)$ are independent, then $\min\{S, T\} \sim \text{Exp}(\lambda + \mu)$. This problem looks at $\max\{S, T\}$.

(a) Use the fact that $\max\{S, T\} \leq t$ if and only if both $S \leq t$ and $T \leq t$ to find the distribution function $\mathbf{P}(\max\{S, T\} \leq t)$.

(b) Use formula (2) to compute $\mathbf{E}[\max\{S, T\}]$.

(c) It is always true that $\min\{S, T\} + \max\{S, T\} = S + T$, since if one of S, T is the minimum then the other must be the maximum. Verify that $\mathbf{E}[\min\{S, T\} + \max\{S, T\}] = \mathbf{E}[S + T]$ by evaluating both sides separately.

3. The purpose of this problem is to prove the *convolution formula*: If X and Y are independent real-valued random variables with density functions f_X and f_Y, then the density function of $X + Y$ is

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(s) f_Y(t - s)\, ds.$$

Let F_X, F_Y, F_{X+Y} be the distribution functions of X, Y, and $X + Y$ respectively. That is, $F_X(c) = \mathbf{P}(X \leq c)$.

(a) Starting from

$$F_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(s)\, \mathbf{P}(X + Y \leq t \mid X = s)\, ds,$$

write $\mathbf{P}(X + Y \leq t \mid X = s)$ in terms of the function F_Y.

(b) If

$$g(r) = \int_{-\infty}^{\infty} f_X(s) f_Y(r - s)\, ds,$$

prove that $\int_{-\infty}^{t} g(r)\, dr = F_{X+Y}(t)$ by switching the order of integration. Explain why this implies that $f_{X+Y}(t) = g(t)$.

4. Start thinking about your project topic:

http://www.math.cornell.edu/~jerison/math4740/project.html

The first draft is due on Friday, April 15 and the final draft is due on Friday, April 29. If you have an idea for a topic, feel free to email me or ask about it in office hours.