Math 4740: Homework 8 Solutions: Extra Credit

Solution 1: The number of satellites that were launched between times 0 and t is $N(t) \sim \text{Poisson}(\lambda t)$. In addition, conditioned on $N(t) = k$, the launch times have the same distribution as k independent uniform random variables on $[0, t]$. Therefore we can re-imagine the launching process as follows. There is a numbered list of satellites, $S_1, S_2, \ldots$. Each satellite picks a launch time independently and uniformly in $[0, t]$. Let U_n be the launch time chosen by satellite S_n. Then, the number of launches $N(t)$ is chosen as an independent Poisson(λt) random variable. Satellites $S_1, S_2, \ldots, S_{N(t)}$ get launched at times $U_1, U_2, \ldots, U_{N(t)}$, and the rest stay on the ground.

Let Y be the lifetime distribution of each satellite. That is, $\mathbf{P}(Y \leq s) = F(s)$, and the probability that any given satellite's lifetime exceeds s is $\mathbf{P}(Y > s) = 1 - F(s)$. Given that satellite S_n is launched at time U_n, the probability that it will still be working at time t is $\mathbf{P}(Y > t - U_n)$. Therefore the overall probability that satellite S_n is still working at time t, averaged over all the possible values of U_n, is

$$p(t) = \frac{1}{t} \int_0^t \mathbf{P}(Y > t - u) du = \frac{1}{t} \int_0^t \mathbf{P}(Y > s) ds.$$

(Change of variables: $s = t - u$.) Not only is each satellite's probability of working at time t equal to $p(t)$, the probabilities are independent for the different satellites. This would not be true if we had listed the satellites in order of launch time, since satellites launched later would be more likely to survive. But the way we have set things up, the survival probabilities are indeed independent.

Let $X(t)$ be the number of satellites that were launched and are still working at time t. We compute $X(t)$ by first computing $N(t) \sim \text{Poisson}(\lambda t)$, the number of launched satellites, and then saying that each one survives with independent probability $p(t)$. This is an example of *thinning*, see section 2.4.1, and the result proved in class says that $X(t) \sim \text{Poisson}(p(t) \cdot \lambda t)$:

$$X(t) \sim \text{Poisson}\left(\lambda \int_0^t \mathbf{P}(Y > s) ds\right) = \text{Poisson}\left(\lambda \int_0^t [1 - F(s)] ds\right).$$

As $t \to \infty$, the Poisson rate converges to $\lambda \int_0^\infty \mathbf{P}(Y > s) ds = \lambda\, \mathbf{E}[Y] = \lambda \mu$ by additional problem 1 on HW 7.

Solution 2: This is the solution given in the textbook, see Theorem 2.12 and Example 2.4 in section 2.4.1. The satellite process works by taking the rate λ Poisson process $\{N(s) : 0 \le s \le t\}$ and keeping a satellite that was launched at time s with probability $1 - F(t - s)$. By Theorem 2.12, the result is a nonhomogeneous Poisson process $\{M(s)\}$ with rate $\tilde{\lambda}(s) = \lambda[1 - F(t - s)]$. If $X(s)$ is the number of satellites still working at time s, then in general $M(s)$ is not necessarily equal to $X(s)$. This is because the definition of $M(s)$ means that if a satellite is *going* to stop working before time t, it is erased immediately. Nevertheless it is true that $M(t) = X(t)$.

The formula for the distribution at time s of a nonhomogeneous Poisson process is $M(s) \sim \text{Poisson}(\int_0^s \tilde{\lambda}(r)dr)$, see p.103 of the printed textbook or p.85 of the PDF. Therefore,

$$X(t) \sim \text{Poisson}\left(\int_0^t \lambda[1 - F(t - r)]dr\right) = \text{Poisson}\left(\lambda \int_0^t [1 - F(s)]ds\right)$$

using the change of variables $s = t - r$. This matches the formula on the previous page, and the rest of the solution is the same.

In the end you can have your pick between the two solutions. Solution 2 is more straightforward once you know the theory of nonhomogeneous Poisson processes, and it is the one the textbook intended. Solution 1 uses only ideas that we have covered in class or on the homework but combines them in a clever way.