MATH 4740 HW8 Solution.

Textbook Exercises:

2.18. To find $\mathbb{P}(X=1)$, suppose:

①. $X \sim \text{Binomial}(n=20, p=0.1)$

$$\Rightarrow \mathbb{P}(X=1) = \binom{20}{1} 0.1^1 \cdot 0.9^{20-1} = \underline{0.27017}.$$

②. $X \sim \text{Poisson}(\lambda = np = 20 \cdot 0.1)$

$$\Rightarrow \mathbb{P}(X=1) = e^{-2} \cdot \frac{2^1}{1!} = \underline{0.27067}$$

Comparing this result to the one from ①, we see that Poisson is not a bad approximation to Binomial.

$\square$.

2.22

We know that $N(t) \sim \text{Poisson}(3t)$

$T_n \sim \text{Gamma}(n, 3)$, $t_n \sim \text{exponential}(3)$

(a). $\mathbb{E}(T_{12}) = \frac{12}{3} = 4$.

(b). By the memoryless property of t_n,

$$\mathbb{E}[T_{12} \mid N(2) = 5] = \mathbb{E}\left[\sum_{i=1}^{12} t_i \;\middle|\; N(2)=5\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{12-5} t_i\right] + 2 \qquad \left(\text{technically, } \mathbb{E}\left[\sum_{i=6}^{12} t_i\right] + 2\right)$$

$$= 7 \cdot \frac{1}{3} + 2$$

$$= \frac{13}{3}$$

$$E[N(5) \mid N(2)=5]$$

$$= E[N(5)-N(2)+N(2)-N(0) \mid N(2)-N(0)=5]$$

$$= E[N(5)-N(2)] + E[N(2)-N(0) \mid N(2)-N(0)=5] \quad \left(\text{independent increments}\right)$$

$$= E[N(5)] - E[N(2)] + 5$$

$$= 3\cdot 5 - 2\cdot 3 + 5$$

$$= 14$$

$\square$.

2.34

Denote the weight of i-th trout by Y_i, the number of trouts caught by time t by $N(t)$. Then

$$N(t) \sim \text{Poisson}(3t).$$

In 2 hours, Edwin is expected to catch $E(N(2)) = 6$ trouts.

So $E\left(\sum_{i=1}^{N(2)} Y_i\right) = E[N(2)]\, E[Y_i]$

$$= 6\cdot 4.$$

$$= 24$$

$$\text{Var}\left(\sum_{i=1}^{N(2)} Y_i\right) = E[N(2)]\,\text{var}(Y_i) + \text{var}(N(2))\,E(Y_i)^2$$

$$= 6\cdot 2^2 + 6\cdot 4^2$$

$$= 120.$$

$$\text{sd}\left(\sum_{i=1}^{N(2)} Y_i\right) = \sqrt{120} = 2\sqrt{30}.$$

$\square$

2.42 Denote the number of trucks passed by time t as $N_T(t)$

the number of cars passed by time t as $N_C(t)$.

$$N_T(t) \sim \text{Poisson}\left(\frac{2}{3}\cdot 0.1 \cdot t\right)$$

$$N_C(t) \sim \text{Poisson}\left(\frac{2}{3}\cdot 0.9 \cdot t\right) \qquad N_C(t) \perp\!\!\!\perp N_T(t).$$

(a) $\mathbb{P}(N_T(60) \geq 1)$

$= 1 - \mathbb{P}(N_T(60) = 0)$

$= 1 - e^{-4}\cdot \frac{4^0}{0!}$

$= 1 - e^{-4}$

(b) $\mathbb{E}\left[N_T(60) + N_C(60) \mid N_T(60) = 10\right]$

$= \mathbb{E}[N_C(60)] + 10$

$= \frac{2}{3}\cdot 0.9 \cdot 60 + 10$

$= 46.$

(c) $\mathbb{P}(N_T(60) = 5,\ N_C(60) = 45 \mid N_T(60) + N_C(60) = 50)$

$= \binom{50}{5} 0.1^5\ 0.9^{45}$

$= 0.1849$

$\square$

2.57

(a) $\mathbb{P}(N(3)=4 \mid N(1)=1)$

$= \mathbb{P}(N(3)-N(1)+N(1)-N(0)=4 \mid N(1)-N(0)=1)$

$= \mathbb{P}(N(3)-N(1)=4-1)$

$= \mathbb{P}(N(2)=3)$

$= e^{-2\cdot 2} \cdot \frac{(2\cdot 2)^3}{3!}$

$= 0.195$

(b) $\mathbb{P}(N(1)=1 \mid N(3)=4)$

$= \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$

$= 0.395$

$\square$

Additional Problems:

1. A Poisson Process $\{N(s)\}$ with rate λ is a stochastic process constructed by a sequence of independent, continuously-occuring events where $N(t) \sim \text{Poisson}(\lambda t)$ and $N(t)$'s are related via ① $N(t)-N(s) \sim \text{Poisson}(\lambda(t-s))$ for all $t>s$ and ② independent increments; whilst a Poisson random variable $N(s)$ is just a random variable, the value of which follows a probability measure as in Poisson distribution,

For any point in the 4-dimensional cube of side length t,

2. (a) the ordering of (t_1, t_2, t_3, t_4) has $4! = 24$ possibilities with equal probability, for example

$$\mathbb{P}(t_1 < t_2 < t_3 < t_4) = \mathbb{P}(t_2 < t_1 < t_4 < t_3) = \cdots = \frac{1}{24}$$

So the space $\{(t_1, t_2, t_3, t_4) \in \mathbb{R}^4 : 0 \le t_1 \le t_2 \le t_3 \le t_4 \le t\}$

has the same volume as $\{(t_1, t_2, t_3, t_4) \in \mathbb{R}^4 : 0 \le t_2 \le t_1 \le t_4 \le t_3 \le t\}$

and all other 22 subspaces.

By symmetry, the volume of any such subspace is $\frac{t^4}{24}$.

(b) If the given joint density is correct, then

$$\int_A f(t_1, t_2, t_3, t_4)\, dt_1\, dt_2\, dt_3\, dt_4$$

$$= \int_A \lambda^4 e^{-\lambda t}\, dt_1\, dt_2\, dt_3\, dt_4$$

$$= \lambda^4 e^{-\lambda t} \cdot \text{vol}(A)$$

$$= e^{-\lambda t} \frac{(\lambda t)^4}{4!}$$

However, the integral should be equal to 1 since it's integrating over a density function. The problem with the given joint density is that it needs to be divided by $\mathbb{P}(N(t)=4)$ because we are conditioning the probability on $N(t)=4$. So the correct joint density is given by: $f(t_1, t_2, t_3, t_4) = \dfrac{\lambda^4 \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}} \cdot \frac{(\lambda t)^4}{4!}} = \dfrac{4!}{t^4}$

(c) Since $U_i \sim$ uniform $[0,t]$, we can view (U_1, U_2, U_3, U_4) as a random point within the 4-dimensional cube of side length 4 as described in (a). Therefore the joint density would be $\frac{1}{\mathrm{vol}(A)} = \frac{4!}{t^4}$

And we know that the joint density of (T_1, T_2, T_3, T_4) Conditioned on $N(t)=4$ is given by $\frac{4!}{t^4}$ as in part (b), which is the same density function of V_i's.

$\square$

3.(a) (i) $M(0) = N(t_0) - N(t_0) = 0$

(ii)
$$\begin{aligned} M(t+s) - M(s) &= \big(N(t_0) - N(t_0-t-s)\big) - \big(N(t_0) - N(t_0-s)\big) \\ &= N(t_0-s) - N(t_0-t-s) \\ &= N(t_0-s-t+t) - N(t_0-t-s) \\ &\sim \text{Poisson}(\lambda t) \end{aligned}$$

(iii) consider $0 \le s_1 \le s_2 \le \cdots \le s_n \le t_0$

$$\begin{aligned} M(s_{i+1}) - M(s_i) &= N(t_0) - N(t_0 - s_{i+1}) - N(t_0) + N(t_0 - s_i) \\ &= N(t_0 - s_i) - N(t_0 - s_{i+1}) \end{aligned}$$

since $0 \le t_0 - s_n \le t_0 - s_{n-1} \le \cdots \le t_0 - s_2 \le t_0 - s_1 \le t_0$, we have $N(t_0 - s_i) - N(t_0 - s_{i+1})$ are independent for $i=1,\cdots,n$ and conclude $M(s)$ has independent increments too.

(b). By (a) we know that $\{M(s)\}$ is a backward Poisson process,

so $t-L$ is just the first arrival of $M(s)$. Also notice that $L=0$

if there is no arrival by time t, which occurs with probability

$1-\int_0^t x\lambda e^{-\lambda x}dx = e^{-\lambda t}$. Therefore,

$$\mathbb{E}(t-L) = \int_0^t x\lambda e^{-\lambda x}dx + (t-0)\cdot e^{-\lambda t}$$

$$= \frac{1}{\lambda}(1-e^{-\lambda t})$$

As $t \to \infty$, $e^{-\lambda t} \to 0$, $\mathbb{E}(t-L) \to \frac{1}{\lambda}$.