Math 4740: Homework 8

Due Friday, April 8 in class.

Textbook exercises (from section 2.6): 2.18, 2.22, 2.34, 2.42, 2.57.

Additional problems:

1. Explain the difference between a Poisson process $\{N(s)\}$ with rate λ and a Poisson random variable $N(s)$ with rate λs.

2. (a) Fix $t > 0$ and let

$$A = \{(t_1, t_2, t_3, t_4) \in \mathbf{R}^4 : 0 \le t_1 \le t_2 \le t_3 \le t_4 \le t\}.$$

Use a symmetry argument to explain why the volume of A is $t^4/24$.

(b) Let $\{N(s)\}$ be a Poisson process with rate λ, arrival times $0 = T_0$, $T_1, T_2, \ldots$, and interarrival times $\tau_k = T_k - T_{k-1}$. In order to have $N(t) = 4$ and $T_1 = t_1, T_2 = t_2, T_3 = t_3, T_4 = t_4$, it is required that $\tau_1 = t_1$, $\tau_2 = t_2 - t_1$, $\tau_3 = t_3 - t_2$, $\tau_4 = t_4 - t_3$, and $\tau_5 > t - t_4$.

Let $f(t_1, t_2, t_3, t_4)$ be the joint density function of T_1, T_2, T_3, T_4 conditioned on $N(t) = 4$. Since an Exp(λ) variable has density function $\lambda e^{-\lambda t}$, one might guess the following (incorrect!) formula for f:

$$f(t_1, t_2, t_3, t_4) = (\lambda e^{-\lambda t_1})(\lambda e^{-\lambda(t_2 - t_1)})(\lambda e^{-\lambda(t_3 - t_2)})(\lambda e^{-\lambda(t_4 - t_3)})(e^{-\lambda(t - t_4)})$$

on A, and $f = 0$ outside A. (The last term is the probability that $\tau_5 > t - t_4$.)

If f were defined by the formula above, what would be

$$\int_A f(t_1, t_2, t_3, t_4)\, dt_1\, dt_2\, dt_3\, dt_4\, ?$$

What *should* that integral be? What was the problem with the formula for f, and how should it be fixed?

(c) Let U_1, U_2, U_3, U_4 be independent uniform random variables on $[0, t]$, and let $V_1 \le V_2 \le V_3 \le V_4$ be the U_i listed in increasing order. Use part (a) to find the joint density function of the V_i, and check that it is the same as the joint density function of the T_i conditioned on $N(t) = 4$.

3. (a) Let $\{N(s)\}$ be a Poisson process with rate λ. Fix $t_0 > 0$ and let $M(s) = N(t_0) - N(t_0 - s)$ for $0 \leq s \leq t_0$. You could think of $\{M(s) : 0 \leq s \leq t_0\}$ as a "backwards Poisson process." Show that $\{M(s)\}$ satisfies all three conditions of Theorem 2.7 in section 2.2, so in fact it is the beginning of a (regular) Poisson process with rate λ.

(b) Use part (a) to solve Exercise 2.29 in the textbook.

Extra credit: Textbook exercise 2.53.