MATH 4740 HW9 Solution

1. (a) $M_{n+1} - M_n = \left(S_{n+1}^2 - (n+1)\right) - \left(S_n^2 - n\right)$

$$= (S_n + X_{n+1})^2 - n - 1 - S_n^2 + n$$

$$= 2S_n X_{n+1} + X_{n+1}^2 - 1$$

(b). ①. $\mathbb{E}[|M_n|] = \mathbb{E}[|S_n^2 - n|]$

$$\leq \mathbb{E}[S_n^2 + n]$$

$$\leq \mathbb{E}[(S_0 + n)^2 + n]$$

$$< \infty \quad \text{for any } n$$

②. $M_n \in \mathcal{A}(M_0, X_1, X_2, \cdots, X_n)$

③. $\mathbb{E}[M_{n+1} - M_n \mid (M_0, M_1, \cdots, M_n)]$

$$= \mathbb{E}[2S_n X_{n+1} + X_{n+1}^2 - 1 \mid (M_0, \cdots, M_n)]$$

$$= 2S_n \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+1}^2) - 1$$

$$= 0 + 1 - 1$$

$$= 0$$

Note: $\mathbb{E}(X_n) = 1 \cdot \left(\frac{1}{2}\right) + (-1) \cdot \frac{1}{2} = 0$

$\mathbb{E}(X_n^2) = 1^2 \cdot \left(\frac{1}{2}\right) + (-1)^2 \cdot \frac{1}{2} = 1$

$$\mathbb{E}(X_n^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$$

$\square$

2. (a) $\mathbb{E}[S_T^2] = 0^2 \cdot \frac{N-x}{N} + N^2 \cdot \frac{x}{N} = Nx$

(b) $\mathbb{E}[M_T] = \mathbb{E}[S_T^2 - T]$

$= \mathbb{E}[S_T^2] - \mathbb{E}[T]$

$= Nx - \mathbb{E}(T)$

Since $\mathbb{E}(M_T) = \mathbb{E}(M_0) = S_0^2 = x^2$

then $\mathbb{E}(T) = Nx - x^2$

$\square$

3. (a) $\mathbb{E}[S_T^2 \,;\, T>n] = \mathbb{P}(T>n)\,\mathbb{E}(S_T^2 \mid T>n)$

$\Big($Since $\mathbb{E}(S_T^2 \mid T>n) = N^2 \cdot \mathbb{P}(S_T^2 = N \mid T>n) + 0^2 \cdot \mathbb{P}(S_T^2 = 0 \mid T>n)$

$\leq N^2\Big)$

$\leq \mathbb{P}(T>n) \cdot N^2$

$\xrightarrow{n\to\infty} 0$ Since $\mathbb{P}(T>\infty) = 0$ by recurrence.

$\mathbb{E}[S_{T\wedge n}^2 \,;\, T>n] = \mathbb{P}(T>n) \cdot \mathbb{E}[S_{T\wedge n}^2 \mid T>n]$

$= \mathbb{P}(T>n)\,\mathbb{E}(S_n^2 \mid T>n)$

$\Big($Since $\mathbb{E}(S_n^2 \mid T>n) = \sum_{k=1}^{N-1} k^2 \cdot \mathbb{P}(S_n^2 = k^2 \mid T>n)$

$\leq \sum N^2 \cdot \mathbb{P}(S_n^2 = k^2 \mid T>n)$

$= N^2\Big)$

$\leq \mathbb{P}(T>n) \cdot N^2$

$\to 0$

(b) $\mathbb{E}[T \wedge n] = \sum_{k=1}^{\infty} \mathbb{P}(T \wedge n \geq k)$

$$= \sum_{k=1}^{n} \mathbb{P}(T \wedge n \geq k) + \sum_{k=n+1}^{\infty} \mathbb{P}(T \wedge n \geq k)$$

$$= \sum_{k=1}^{n} \mathbb{P}(T \geq k) + 0$$

$$= \sum_{k=1}^{n} \mathbb{P}(T \geq k)$$

$$\xrightarrow{n \to \infty} \mathbb{E}(T).$$

$\square$

4. (a) $L_{n+1} - L_n = (S_{n+1}^3 - a(n+1)S_{n+1}) - S_n^3 + anS_n$

$$= \left[(S_n + X_{n+1})^3 - a(n+1)(S_n + X_{n+1})\right] - S_n^3 + anS_n$$

$$= X_{n+1}^3 + 3S_n X_{n+1}^2 + \left[3S_n^2 - a(n+1)\right] X_{n+1} - aS_n$$

$$\mathbb{E}\left[L_{n+1} - L_n \mid (L_0, X_1, \dots, X_n)\right]$$

$$= \mathbb{E}\left(X_{n+1}^3 + 3S_n X_{n+1}^2 + X_{n+1}(3S_n^2 - a(n+1)) - aS_n \mid (L_0, X_1, \dots, X_n)\right]$$

$$= 0 + 3S_n + 0 - aS_n$$

$$= 0$$

$$\Rightarrow a = 3.$$

(b) $\mathbb{E}[L_0] = S_0^3 = x^3$

$$\mathbb{E}[L_T] = \mathbb{P}(S_T = 0)\,\mathbb{E}[L_T \mid S_T = 0] + \mathbb{P}(S_T = N)\,\mathbb{E}[L_T \mid S_T = N]$$

$$= \frac{N-x}{x} \cdot 0 + \frac{x}{N} \cdot \left(N^3 - a\,\mathbb{E}[T \mid S_T = N] \cdot N\right)$$

$$= xN^2 - 3x\,\mathbb{E}(T \mid S_T = N)$$

$$\mathbb{E}(L_T) = \mathbb{E}(L_0) \Rightarrow \mathbb{E}[T \mid S_T = N] = \frac{N^2 - x^2}{3}$$

(c) $\mathbb{E}[L_T] = \mathbb{P}(S_T=0)\cdot \mathbb{E}[L_T \mid S_T=0] + \mathbb{P}(S_T=N)\,\mathbb{E}[L_T \mid S_T=N]$

$$= \frac{x}{N}\cdot 0 + \frac{N-x}{N}\cdot\left(N^3 - a\cdot N\,\mathbb{E}[T \mid S_T=N]\right)$$

$$= (N-x)N^2 - 3(N-x)\,\mathbb{E}[T \mid S_T=N]$$

$$\mathbb{E}[L_T] = \mathbb{E}[L_0] = (N-x)^3 \Rightarrow \mathbb{E}[T \mid S_T=N] = \frac{2NX-x^2}{3}$$

which implies that when $S_0 = x$, $\mathbb{E}[L_0] = x^3$,

$$\mathbb{E}[T \mid S_T=0] = \frac{2Nx-x^2}{3}$$

Hence

$$RHS = \mathbb{P}(S_T=0)\,\mathbb{E}(T \mid S_T=0) + \mathbb{P}(S_T=N)\,\mathbb{E}[T \mid S_T=N]$$

$$= \frac{N-x}{N}\cdot\frac{2Nx-x^2}{3} + \frac{x}{N}\cdot\frac{N^2-x^2}{3}$$

$$= Nx - x^2$$

$$= \mathbb{E}(T)$$

$$= LHS.$$

$\square$