Math 4740: Homework 9

Due Friday, April 22 in class.

1. Let $X_1, X_2, \ldots$ be iid with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = 1/2$, and let $S_n = S_0 + X_1 + X_2 + \cdots + X_n$ be a simple random walk on $\mathbf{Z}$ started at S_0.

(a) Let $M_n = S_n^2 - n$. Verify that $M_{n+1} - M_n = 2S_nX_{n+1} + X_{n+1}^2 - 1$.

(b) Argue using part (a) that (M_n) is a martingale.

2. In the setting of Problem 1, fix $N > 0$ and suppose that $S_0 = x$ for some $0 < x < N$. Let $T = \min\{n \geq 0 : S_n = 0 \text{ or } S_n = N\}$. This is the number of bets the Gambler's Ruin gambler must make before either winning $\$N$ or losing everything.

(a) It was proved in class (twice in fact, once in Chapter 1 and once in Chapter 5) that $\mathbf{P}(S_T = 0) = (N - x)/N$, meaning that $\mathbf{P}(S_T = N) = x/N$. Use this to compute $\mathbf{E}[S_T^2]$.

(b) Assuming that $\mathbf{E}[M_T] = \mathbf{E}[M_0]$, use part (a) to find a formula for $\mathbf{E}[T]$.

3. This problem justifies the assumption that $\mathbf{E}[M_T] = \mathbf{E}[M_0]$ in Problem 2. As shown in class, it suffices to show that

$$\mathbf{E}[M_T] = \lim_{n\to\infty} \mathbf{E}[M_{T\wedge n}].$$

This will be done in two parts.

(a) The first part is to show that

$$\mathbf{E}[S_T^2] = \lim_{n\to\infty} \mathbf{E}[S_{T\wedge n}^2]. \tag{1}$$

Since $|S_{T\wedge n}^2| \leq N^2$ (and $|S_T^2| \leq N^2$), we can use the argument given in class: Decompose

$$\mathbf{E}[S_T^2] = \mathbf{E}[S_T^2; T \leq n] + \mathbf{E}[S_T^2; T > n],$$
$$\mathbf{E}[S_{T\wedge n}^2] = \mathbf{E}[S_T^2; T \leq n] + \mathbf{E}[S_{T\wedge n}^2; T > n],$$

and argue that both of the second terms tend to 0 as $n \to \infty$. Fill in the details of this argument.

(b) The second part is to show that

$$\mathbf{E}[T] = \lim_{n\to\infty} \mathbf{E}[T \wedge n]. \tag{2}$$

We know that

$$\mathbf{E}[T] = \sum_{k=1}^{\infty} \mathbf{P}(T \geq k), \qquad \mathbf{E}[T \wedge n] = \sum_{k=1}^{\infty} \mathbf{P}(T \wedge n \geq k).$$

Prove the desired statement by simplifying the expression $\mathbf{P}(T \wedge n \geq k)$ when $k \leq n$ and when $k > n$.

The conclusion $\mathbf{E}[M_T] = \lim_{n\to\infty} \mathbf{E}[M_{T\wedge n}]$ follows by taking the difference (1) − (2).

4. (a) Still in the setting of Problems 1–3, solve for the value of a that makes $L_n = S_n^3 - anS_n$ a martingale.

(b) Let T be as in the previous problems. Assuming that $\mathbf{E}[L_T] = \mathbf{E}[L_0]$, compute the conditional expectation $\mathbf{E}[T \mid S_T = N]$ when $S_0 = x$.

(c) By a symmetry argument, the value of $\mathbf{E}[T \mid S_T = 0]$ when $S_0 = x$ is the same as the value of $\mathbf{E}[T \mid S_T = N]$ when $S_0 = N - x$. Use this fact and part (b) to compute $\mathbf{E}[T \mid S_T = 0]$ when $S_0 = x$. Finally, verify that

$$\mathbf{E}[T] = \mathbf{P}(S_T = 0)\,\mathbf{E}[T \mid S_T = 0] + \mathbf{P}(S_T = N)\,\mathbf{E}[T \mid S_T = N]$$

when $S_0 = x$ by plugging in the formulas you have found for all five quantities in the equation and checking that it is correct.