Math 4740: Prelim Solutions

March 11, 2015

1. (18 points) Let (X_n) be a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition matrix given below.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(a) (3 points) Find all the communicating classes of recurrent states.

Draw the graph showing which states can reach each other in a single step.

The communicating classes of recurrent states are $\{a, b\}, \{e\}, \{f\}$.

(b) (4 points) Characterize all the stationary distributions of the chain.

For each communicating class of recurrent states, there is a unique stationary distribution supported on that class. The restriction of P to $C_1 = \{a, b\}$ is

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix},$$

which has stationary distribution $\begin{bmatrix} \frac{0.2}{0.2+0.6} & \frac{0.6}{0.2+0.6} \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix}$. The associated stationary distribution of P is $\pi_1 = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 & 0 \end{bmatrix}$. For $C_2 = \{e\}$ and $C_3 = \{f\}$ we have

$$\pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \pi_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The general form for a stationary distribution of P is $\pi = r_1\pi_1 + r_2\pi_2 + r_3\pi_3$ where $r_1 + r_2 + r_3 = 1$ and $r_1, r_2, r_3 \geq 0$. This can also be written as

$$\pi = \begin{bmatrix} (1/4)r_1 & (3/4)r_1 & 0 & 0 & r_2 & r_3 \end{bmatrix}.$$

(c) (4 points) For any subset A of the state space, let $V_A = \min\{n \geq 0 : X_n \in A\}$. Compute $\mathbf{P}_c(V_{\{e,f\}} < \infty)$.

Let $h(x) = \mathbf{P}_x(V_{\{e,f\}} < \infty)$, then $h(a) = h(b) = 0$ and $h(e) = h(f) = 1$. For $x \in \{c, d\}$ we have

$$h(x) = \sum_y P(x,y)h(y)$$

which gives the equations

$$\begin{aligned} h(c) &= 0.4h(b) + 0.6h(d) = 0.6h(d), \\ h(d) &= 0.5h(c) + 0.3h(e) + 0.2h(f) = 0.5h(c) + 0.5. \end{aligned}$$

Plugging the second equation into the first yields

$$h(c) = 0.6[0.5h(c) + 0.5] = 0.3h(c) + 0.3 \Rightarrow 0.7h(c) = 0.3 \Rightarrow h(c) = 3/7.$$

(d) (4 points) Compute $\mathbf{E}_c[V_{\{a,b,e,f\}}]$.

Let $g(x) = \mathbf{E}_x[V_{\{a,b,e,f\}}]$, then $g(a) = g(b) = g(e) = g(f) = 0$. For $x \in \{c, d\}$ we have

$$g(x) = 1 + \sum_y P(x,y)g(y)$$

which gives the equations

$$\begin{aligned} g(c) &= 1 + 0.4g(b) + 0.6g(d) = 1 + 0.6g(d), \\ g(d) &= 1 + 0.5g(c) + 0.3g(e) + 0.2g(f) = 1 + 0.5g(c). \end{aligned}$$

Plugging the second equation into the first yields

$$g(c) = 1 + 0.6[1 + 0.5g(c)] = 1.6 + 0.3g(c) \Rightarrow 0.7g(c) = 1.6 \Rightarrow g(c) = 16/7.$$

(e) (3 points) Compute $\lim_{n\to\infty} P^n(c, a)$.

By the main convergence theorem for the chain restricted to $\{a, b\}$, if the random walker enters $\{a, b\}$ its limiting probability of being at state a is $\pi_1(a) = 1/4$. Given that the walker starts at c, its probability of entering $\{a, b\}$ is $\mathbf{P}_c(V_{\{a,b\}} < \infty) = 1 - \mathbf{P}_c(V_{\{e,f\}} < \infty) = 4/7$ from part (c). Therefore

$$\lim_{n\to\infty} P^n(c, a) = \mathbf{P}_c(V_{\{a,b\}} < \infty) \cdot \pi_1(a) = \frac{4}{7} \cdot \frac{1}{4} = \frac{1}{7}.$$

2. (16 points) Adrian, Betty, and Christine are three friends trying to decide whether to support Candidate Red or Candidate Blue in the upcoming election. At any given time, each of them has a preference, but they change their minds frequently. Every morning, exactly one of the three friends (chosen with probability 1/3 each) forgets his or her current preference and selects a new candidate by flipping a fair coin, while the other two friends keep their preferences that day.

If you use a Markov chain or transition matrix when solving this problem, make sure to define explicitly all your notation.

(a) (3 points) Right now, all three friends support Candidate Red. What is the probability that two days from now, at least one of them will support Candidate Blue?

Let X_n be the number of friends supporting Candidate Blue on day n. This is a Markov chain on $\{0, 1, 2, 3\}$. Let P be its transition matrix.

We compute: $P(0,0) = \frac{1}{2}$, $P(0,1) = \frac{1}{2}$, $P(1,0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. Therefore

$$P^2(0,0) = P(0,0)P(0,0) + P(0,1)P(1,0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{3}.$$

The problem is asking for $1 - P^2(0,0) = \frac{2}{3}$.

(b) (6 points) What is the expected number of days from now until the next time that all three friends support Candidate Red?

This is $\mathbf{E}_0[T_0] = 1/\pi(0)$ where π is the stationary distribution. To compute π we first finish writing the transition matrix.

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & 1/6 & 1/2 & 1/3 & 0 \\ 2 & 0 & 1/3 & 1/2 & 1/6 \\ 3 & 0 & 0 & 1/2 & 1/2 \end{array}$$

Since this is a birth and death chain, the detailed balance equations must hold: $\pi(i)P(i, i+1) = \pi(i+1)P(i+1, i)$ for $i = 0, 1, 2$. Thus

$$\pi(0) \cdot \frac{1}{2} = \pi(1) \cdot \frac{1}{6}, \qquad \pi(1) \cdot \frac{1}{3} = \pi(2) \cdot \frac{1}{3}, \qquad \pi(2) \cdot \frac{1}{6} = \pi(3) \cdot \frac{1}{2}$$

which gives $\pi(1) = 3\pi(0)$, $\pi(2) = \pi(1) = 3\pi(0)$, $\pi(3) = \frac{1}{3}\pi(2) = \pi(0)$. Now

$$1 = \pi(0) + \pi(1) + \pi(2) + \pi(3) = \pi(0)(1 + 3 + 3 + 1) \Rightarrow \pi(0) = \frac{1}{8}.$$

So the answer to the question is $\mathbf{E}_0[T_0] = 1/\pi(0) = 8$ days.

Note: Some students let $g(i) = \mathbf{E}_i[T_0]$ and got the system of equations

$$g(0) = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot g(1),$$
$$g(1) = 1 + \frac{1}{6} \cdot 0 + \frac{1}{2} \cdot g(1) + \frac{1}{3} \cdot g(2),$$

and so forth. This approach does work. It takes longer but also gives extra information, namely the values of $g(i)$ for $i \neq 0$.

(c) (3 points) On election day (200 days from now) the three friends will take an informal vote amongst themselves. What is the approximate probability that the vote will be unanimous in favor of one candidate?

From part (b), $\pi = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$. The distribution of X_{200} is not exactly π, but very close (by the main convergence theorem). The probability of unanimity is

$$\mathbf{P}_0(X_{200} \in \{0, 3\}) \approx \pi(0) + \pi(3) = \frac{1}{4}.$$

(d) (4 points) The three friends meet every day for lunch. If they all support one candidate that day, they make a group donation of \$10 to that candidate's campaign. If two friends support one candidate while the third friend supports the other, they make a group donation of \$5 to the candidate preferred by the majority. If they meet for lunch 200 times, approximately how much money in total will they donate to each candidate?

Define functions f, g on $\{0, 1, 2, 3\}$ by $f(0) = 10$, $f(1) = 5$, $f(2) = 0$, $f(3) = 0$ and $g(0) = 0$, $g(1) = 0$, $g(2) = 5$, $g(3) = 10$. Using the ergodic theorem, the total amount donated to Candidate Red is

$$\sum_{n=1}^{200} f(X_n) \approx 200 \sum_{i=0}^{3} \pi(i) f(i) = 200 \left(\frac{1}{8} \cdot 10 + \frac{3}{8} \cdot 5 \right) = 625.$$

For Candidate Blue one can set up a similar sum with g, or simply argue by symmetry that the total amount should also be about \$625.

3. (18 points; 3 each) **True/False.** Consider a Markov chain on a finite state space $\mathcal{X}$ with transition matrix P. Assume that it has a **unique** stationary distribution π.

For each statement, circle "True" if the statement is *necessarily* true, and "False" if the statement *could possibly* be false. **You do not need to explain your answers; simply circle "True" or "False."**

(a) The Markov chain is aperiodic. Answer: **False**. A counterexample is the deterministic chain that always moves in one direction around a cycle, for example $\mathcal{X} = \{1, 2, 3\}$ and $P(1, 2) = P(2, 3) = P(3, 1) = 1$. This has unique stationary distribution $\pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ despite being periodic.

(b) The Markov chain is irreducible. Answer: **False**. It is possible for there to be transient states. For example: $\mathcal{X} = \{1, 2, 3\}$, $P(1, 2) = 1$, $P(2, 2) = P(2, 3) = \frac{1}{2}$, $P(3, 2) = P(3, 3) = \frac{1}{2}$. This has unique stationary distribution $\pi = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ despite being reducible. However, if we assumed that all states were recurrent (or equivalently that $\pi(x) > 0$ for all x), the chain would necessarily be irreducible.

(c) There is exactly one communicating class of recurrent states. Answer: **True**. For finite Markov chains this is equivalent to the condition that π is unique; Problem 1 shows what happens when there is more than one communicating class of recurrent states.

(d) If P is a symmetric matrix, then $\pi(x) = \pi(y)$ for all $x, y \in \mathcal{X}$. Answer: **True**. Suppose there are N states, and let $\mu = \begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}$ be the uniform distribution on the state space. The condition that $P(x, y) = P(y, x)$ implies that $\mu(x)P(x, y) = \mu(y)P(y, x)$, that is, the chain satisfies the detailed balance condition with respect to μ. Therefore μ is a stationary distribution, and since π is unique, $\mu = \pi$.

(e) If $\pi(x) = \pi(y)$ for all $x, y \in \mathcal{X}$, then P is a symmetric matrix. Answer: **False**. The deterministic cycle in part (a) is also a counterexample here. Another way of looking at it is this. The condition $\pi(x) = \pi(y)$ for all x, y means that the uniform distribution is stationary. If P is symmetric, then the chain satisfies detailed balance with respect to the uniform distribution, which is a stronger condition.

(f) Given $x, y \in \mathcal{X}$, if $\rho_{xy} = \mathbf{P}_x(V_y < \infty) = 1/2$, then x is transient. Answer: **True**. Suppose for contradiction that x is recurrent, and let C_x be its communicating class. If $z \notin C_x$ then $\rho_{xz} = 0$, while if $z \in C_x$ then $\rho_{xz} = 1$. So ρ_{xz} is either 0 or 1 for all $z \in \mathcal{X}$. If there exists y such that $0 < \rho_{xy} < 1$, then x must be transient.