Math 4740: Prelim

March 11, 2015

Name:

NetID:

Instructions: This exam is 50 minutes long. It has 8 pages including the cover and 3 questions worth a total of 52 points. No written or electronic aids are allowed.

Please fully explain all your answers except when explicitly instructed otherwise. If you need more space to answer a question, use the back of the page or the blank sheet at the end of the exam. Label your work clearly if you use extra space.

Academic integrity is expected of all Cornell University students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

Signature:

1:	
2:	
3:	
Total score:	

1. (18 points) Let (X_n) be a Markov chain with state space $\{a, b, c, d, e, f\}$ and transition matrix given below.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(a) (3 points) Find all the communicating classes of recurrent states.

(b) (4 points) Characterize all the stationary distributions of the chain.

(c) (4 points) For any subset A of the state space, let $V_A = \min\{n \geq 0 : X_n \in A\}$. Compute $\mathbf{P}_c(V_{\{e,f\}} < \infty)$.

(d) (4 points) Compute $\mathbf{E}_c[V_{\{a,b,e,f\}}]$.

(e) (3 points) Compute $\lim_{n\to\infty} P^n(c, a)$.

2. (16 points) Adrian, Betty, and Christine are three friends trying to decide whether to support Candidate Red or Candidate Blue in the upcoming election. At any given time, each of them has a preference, but they change their minds frequently. Every morning, exactly one of the three friends (chosen with probability 1/3 each) forgets his or her current preference and selects a new candidate by flipping a fair coin, while the other two friends keep their preferences that day.

If you use a Markov chain or transition matrix when solving this problem, make sure to define explicitly all your notation.

(a) (3 points) Right now, all three friends support Candidate Red. What is the probability that two days from now, at least one of them will support Candidate Blue?

(b) (6 points) What is the expected number of days from now until the next time that all three friends support Candidate Red?

(c) (3 points) On election day (200 days from now) the three friends will take an informal vote amongst themselves. What is the approximate probability that the vote will be unanimous in favor of one candidate?

(d) (4 points) The three friends meet every day for lunch. If they all support one candidate that day, they make a group donation of \$10 to that candidate's campaign. If two friends support one candidate while the third friend supports the other, they make a group donation of \$5 to the candidate preferred by the majority. If they meet for lunch 200 times, approximately how much money in total will they donate to each candidate?

3. (18 points; 3 each) **True/False.** Consider a Markov chain on a finite state space $\mathcal{X}$ with transition matrix P. Assume that it has a **unique** stationary distribution π.

For each statement, circle "True" if the statement is *necessarily* true, and "False" if the statement *could possibly* be false. **You do not need to explain your answers; simply circle "True" or "False."**

(a) The Markov chain is aperiodic. True False

(b) The Markov chain is irreducible. True False

(c) There is exactly one communicating class of recurrent states. True False

(d) If P is a symmetric matrix, then $\pi(x) = \pi(y)$ for all $x, y \in \mathcal{X}$. True False

(e) If $\pi(x) = \pi(y)$ for all $x, y \in \mathcal{X}$, then P is a symmetric matrix. True False

(f) Given $x, y \in \mathcal{X}$, if $\rho_{xy} = \mathbf{P}_x(V_y < \infty) = 1/2$, then x is transient. True False

(This page for scratch work.)