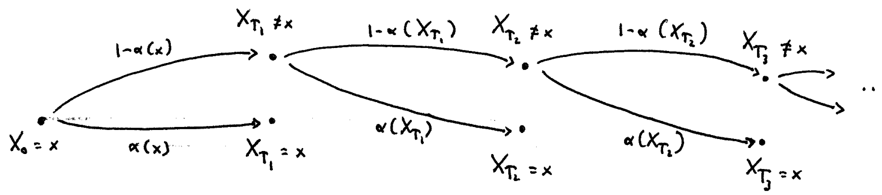
Math 4740: Proof of Theorem 1.7

Theorem. *Let (X_n) be a Markov chain on the state space $\mathcal{X}$ with transition matrix P. If $C \subseteq \mathcal{X}$ is finite, closed, and irreducible, then all states in C are recurrent.*

Proof. Fix $x \in C$. Since C is irreducible, $y \to x$ for every $y \in C$. Therefore, there exists $m(y) \geq 1$ such that $P^{m(y)}(y,x) > 0$. Set $\alpha(y) = P^{m(y)}(y,x)$ and $\alpha = \min\{\alpha(y) : y \in C\}$.

Suppose the Markov chain (X_n) is started from x. We need to prove that the chain returns to x with probability 1. Let $T_1 = m(x)$, so that $\mathbf{P}_x(X_{T_1} = x) = \alpha(x)$ and $\mathbf{P}_x(X_{T_1} \neq x) = 1-\alpha(x) \leq 1-\alpha$. If $X_{T_1} = x$, we are done. If not, then $X_{T_1} \in C$ (since C is closed). Thus, if we run the chain for another $m(X_{T_1})$ steps, there will be a probability of $\alpha(X_{T_1})$ that we reach x at time $T_2 = T_1 + m(X_{T_1})$. The diagram below illustrates this process.



Formally, we define a sequence of stopping times $T_1, T_2, \ldots$ by $T_1 = m(x)$ and for $k \geq 1$, $T_{k+1} = T_k + m(X_{T_k})$. Since each failure probability is $1-\alpha(X_{T_k}) \leq 1-\alpha$, it seems intuitively true that for all $k \geq 1$,

$$\mathbf{P}_x(T_x > T_k) \leq (1-\alpha)^k, \tag{1}$$

where $T_x = \min\{n \geq 1 : X_n = x\}$. Taking the limit as $k \to \infty$ implies that $\mathbf{P}_x(T_x = \infty) = 0$, that is, x is recurrent. Therefore it remains to prove (1). We argue by induction on k.

Base case $k = 1$: $\mathbf{P}_x(T_x > T_1) \leq \mathbf{P}_x(X_{T_1} \neq x) = 1 - P^{m(x)}(x,x) =$

$1-\alpha(x) \le 1-\alpha$. Assume now that (1) holds for k. We compute:

$$\begin{aligned}
&\mathbf{P}_x(T_x > T_{k+1}) \\
&\le \mathbf{P}_x(T_x > T_k, X_{T_{k+1}} \ne x) \\
&= \sum_{\substack{y \in C \\ y \ne x}} \mathbf{P}_x(T_x > T_k, X_{T_k} = y, X_{T_{k+1}} \ne x) \\
&= \sum_{\substack{y \in C \\ y \ne x}} \mathbf{P}_x(T_x > T_k, X_{T_k} = y)\, \mathbf{P}_x(X_{T_{k+1}} \ne x \mid T_x > T_k, X_{T_k} = y).
\end{aligned}$$

Because the conditions $T_x > T_k, X_{T_k} = y$ depend only on the history up to the stopping time T_k, the strong Markov property implies that

$$\mathbf{P}_x(X_{T_{k+1}} \ne x \mid T_x > T_k, X_{T_k} = y) = \mathbf{P}_y(X_{m(y)} \ne x) = 1-\alpha(y) \le 1-\alpha.$$

Therefore,

$$\begin{aligned}
\mathbf{P}_x(T_x > T_{k+1}) &\le \sum_{\substack{y \in C \\ y \ne x}} \mathbf{P}_x(T_x > T_k, X_{T_k} = y)(1-\alpha) \\
&= (1-\alpha)\, \mathbf{P}_x(T_x > T_k) \\
&\le (1-\alpha)(1-\alpha)^k = (1-\alpha)^{k+1},
\end{aligned}$$

using the inductive hypothesis on the last line. This completes the proof. □