Math 4740: American call options

Consider a stock whose price at time n is S_n, and an American option with expiration time N. Assume that the payoff g_n from exercising the option at time n depends only on the stock price S_n: $g_n = g(S_n)$ for some fixed function g. If the option is a call with strike price K, then $g(x) = \max\{0, x - K\}$. If the option is a put with strike price K, then $g(x) = \max\{0, K - x\}$.

As usual, we let r be the interest rate and $\mathbf{P}^*$, $\mathbf{E}^*$ denote probability and expectation with respect to the risk-neutral measure, so that $S_n/(1+r)^n$ is a martingale under $\mathbf{P}^*$. This means that for every possible history $(S_0, \ldots, S_n)$ of stock prices up to time n,

$$\frac{1}{1+r}\,\mathbf{E}^*[S_{n+1} \mid S_0, \ldots, S_n] = S_n.$$

Theorem. *In the situation described above, if g is a convex function and $g(0) = 0$, then*

$$g(S_n) \le \frac{1}{1+r}\,\mathbf{E}^*[g(S_{n+1}) \mid S_0, \ldots, S_n].$$

The left side is the payoff from exercising the option at time n. The right side is the expected payoff from exercising the option at time $n+1$ given the history up to time n, discounted by the interest rate.

Proof. We saw in class that since g is convex and $g(0) = 0$, it follows that for all $0 \le \lambda \le 1$, $g(\lambda x) \le \lambda g(x)$. Therefore,

$$g(S_n) = g\left(\frac{1}{1+r}\,\mathbf{E}^*[S_{n+1} \mid S_0, \ldots, S_n]\right) \le \frac{1}{1+r} g\left(\mathbf{E}^*[S_{n+1} \mid S_0, \ldots, S_n]\right).$$

It remains to show that

$$g\left(\mathbf{E}^*[S_{n+1} \mid S_0, \ldots, S_n]\right) \le \mathbf{E}^*[g(S_{n+1}) \mid S_0, \ldots, S_n].$$

This is a general fact known as *Jensen's inequality*: for any convex function g and any real-valued random variable X, $g(\mathbf{E}[X]) \le \mathbf{E}[g(X)]$. See

http://www.math.uah.edu/stat/expect/Properties2.html#jen for a proof; after reading down to statement 9, click where it says "▶ *Proof:*" to see the actual proof. Note that the definition of convexity given in that link is different from ours, but it is equivalent.

The approach in class was to assume that given S_n, there are only two possibilities for S_{n+1}. What we said was, if a is a sequence of n coin-flips representing the history of the stock price up to time n, then

$$\begin{aligned}\mathbf{E}^*[S_{n+1} \mid a] &= \mathbf{P}^*(aH \mid a)S_{n+1}(aH) + \mathbf{P}^*(aT \mid a)S_{n+1}(aT),\\ \mathbf{E}^*[g(S_{n+1}) \mid a] &= \mathbf{P}^*(aH \mid a)g(S_{n+1}(aH)) + \mathbf{P}^*(aT \mid a)g(S_{n+1}(aT)).\end{aligned}$$

We want to show that g applied to the first line is less than or equal to the second line. This is a statement of the form

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda)g(x_2)$$

where $\lambda = \mathbf{P}^*(aH \mid a)$, $x_1 = S_{n+1}(aH)$, and $x_2 = S_{n+1}(aT)$. Therefore it is true by convexity of g. $\square$

As a consequence of this theorem, we can conclude that if g is convex with $g(0) = 0$ (which holds for the American call), it is always optimal to wait until the expiration time N to exercise the option. The reasoning is this. Suppose we are sitting at time $n < N$ and we have not yet exercised the option. There are three choices:

1. Exercise the option now at time n.
2. Wait and exercise the option at time $n+1$.
3. Wait until time $n+1$ and then re-evaluate according to the optimal strategy given the history through time $n+1$.

The theorem says that choice 2 has a higher expected payoff than choice 1. Choice 3 optimizes over many strategies, one of which is choice 2, so choice 3 has an equal-or-higher expected payoff than choice 2. We conclude that choice 3 will maximize the expected payoff. Therefore we proceed to time $n+1$ without exercising the option. Now apply this reasoning again; it says to proceed to time $n+2$ without exercising the option. We continue up until time N when we have no choice but to exercise the option.