Math 6710, Fall 2016
Homework 13

1. Let $(S,\mathcal{S})$ be a "nice" measurable space (one for which Kolmogorov extension works; for this problem you can imagine $(S,\mathcal{S}) = (\mathbf{R}^d, \mathcal{B}^d)$). Set $\Omega = S^{\mathbf{N}} = \{\omega = (\omega_1, \omega_2, \ldots) : \text{each } \omega_n \in S\}$ with the product σ-algebra $\mathcal{S}^{\mathbf{N}}$ and define random variables X_n by $X_n(\omega) = \omega_n$. It is true (you do not have to show) that $\mathcal{P} = \cup_{n=1}^{\infty} \sigma(X_1, \ldots, X_n)$ is a π-system and that $\sigma(\mathcal{P}) = \sigma(X_1, X_2, \ldots) = \mathcal{S}^{\mathbf{N}}$. Let $\mathcal{L} = \{J \in \mathcal{S}^{\mathbf{N}} : \text{there exist } I_n \in \sigma(X_1, \ldots, X_n) \text{ with } P(I_n \triangle J) \to 0\}$. Prove, as claimed in Lemma 16.1 of the notes, that $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$.

2. Suppose the sequence of random variables $\{Y_n\}$ is Cauchy in L^2, that is, for all $\varepsilon > 0$ there exists N such that for all $m, n \geq N$, $\|Y_m - Y_n\|_2 = E[(Y_m - Y_n)^2]^{1/2} < \varepsilon$. Show that if $Y_n \to Y$ almost surely, then $E[Y_n^2] \to E[Y^2]$. *Hint:* If $Y_n \to Z$ in probability then there is a subsequence Y_{n_m} that converges to Z almost surely.

3. Durrett Exercise 4.1.2.

4. Let S and T be stopping times with respect to the sequence $\{X_n\}$ of random variables.

(a) Show that $S \wedge T$ and $S \vee T$ are also stopping times. (Note, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.) Since $T \equiv n$ is a stopping time, this shows that $S \wedge n$ and $S \vee n$ are stopping times for any n.

(b) Is $S + T$ a stopping time? If $S < T$ always, is $T - S$ a stopping time? In each case provide a proof or a counterexample.

5. Durrett Exercise 4.1.9. It may help to solve Exercise 4.1.8 as a stepping stone; your argument will probably use Theorem 4.1.4 at some point.

6. Durrett Exercise 4.1.13, ignoring the last sentence about Exercise 4.1.10.