Grading: 1, 2, 3, 5 (each 5 pts) and 4, 6 (10 pts).

Problem 1: For any $x \in \mathcal{D}$ we have that $F(x-) < F(x+)$, since F is non-decreasing and discontinuous. Hence, $\exists$ a rational number r s.t. $F(x-) < r < F(x+)$. For each x choose one of such r's and denote it by r_x. So, $F(x-) < r_x < F(x+)$ and for $x \neq y$, $x, y \in \mathcal{D}$, $r_x \neq r_y$ (we can WLOG assume $x < y$ so $r_x < F(x+) \leq F(y-) < r_y$, since F is non-decreasing). Thus, $R(x) := r_x$, $R : \mathcal{D} \to \mathbb{Q}$ is an injective function and $|\mathbb{Q}| \geq |\mathcal{D}|$, which implies that $\mathcal{D}$ is countable, since $\mathbb{Q}$ also is. (Note: One can avoid the Axiom of Choice by reasoning that the number of $x \in \mathcal{D}$ for which $F(x+) - F(x-) \geq 1/k$ is at most k for each positive integer k.)

Problem 2: F is continuous on $\mathbb{R}$ iff for all $x \in \mathbb{R}$ and any monotonic sequence (WLOG choose a non-decreasing) $x_n \nearrow x$ we have $F(x_n) - F(x) \to 0$, which is equivalent to $\mu(A_n) := \mu((x_n, x]) \to 0$, where $A_n := (x_n, x]$. But, since μ is measure, $A_n \supseteq A_{n+1}$ and $\cap_{n=1}^{\infty} A_n = \{x\}$, the latter limit is equivalent to $\mu(\{x\}) = 0$. This completes the proof.

Problem 3: We show that $P(Y \leq y) = y$ for all $y \in (0, 1)$. Once this is done, $P(Y \leq 0) = \lim_{y \searrow 0} P(Y \leq y) = 0$ and $P(Y \leq 1) \geq \lim_{y \nearrow 1} P(Y \leq y) = 1$. So, fix $y \in (0, 1)$. Since F is continuous, by the Intermediate Value Theorem there is at least one $x \in \mathbb{R}$ such that $F(x) = y$. Define $F^{-1}(y) = \sup\{x : F(x) = y\}$. By continuity, $F(F^{-1}(y)) = y$. Since F is nondecreasing, $F(x) \leq y$ iff $x \leq F^{-1}(y)$, so $P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$.

Problem 4: Note that if we denote $P = \{(-\infty, x] : x \in \mathbb{R}\}$ and $D = \{E \in \mathcal{B} : \int_E f dm = \mu(E)\} \subseteq \mathcal{B}$, then by the problem conditions $P \subseteq D$ and we need to prove that $D = \mathcal{B}$. Since the half-open intervals generate $\mathcal{B}$ i.e. $\mathcal{B} = \sigma(P)$, it is enough to show $\sigma(P) \subseteq D$ (because it would imply $\mathcal{B} = \sigma(P) \subseteq D \subseteq \mathcal{B}$). To do that we use Dynkin's $\pi - \lambda$ theorem and we are only left to show that P is a π-system and D is a λ-system. Next, P is a π-system since it is non-empty and for any $x, y \in \mathbb{R}$ s.t. $x \leq y$ we have $(-\infty, x] \cap (-\infty, y] = (-\infty, \min(x, y)] \in P$. Using the properties of measure, we also have that **1)** $\int_{\mathbb{R}} f dm = 1 = \mu(\mathbb{R})$, **2)** for any set $A \in D$, $\int_{A^c} f dm = \int_{\mathbb{R}} f dm - \int_A f dm = \mu(\mathbb{R}) - \mu(A) = \mu(A^c)$ and **3)** for any sequence of disjoint $A_n \in D$, $\int_{\cup A_n} f dm = \sum_n \int_{A_n} f dm = \sum_n \mu(A_n) = \mu(\cup A_n)$. Hence, D satisfies the definition of a λ-system and the proof is complete.

Problem 5: No, take $\Omega = \{0, 1\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{0\}) = P(\{1\}) = \frac{1}{2}$, $X(0) = 0$, $X(1) = 1$, $Y(0) = 1$, $Y(1) = 0$ and $Z = X$. Then $X =_d Y$ but $XZ = X$ and $YZ = 0$ so $XZ \neq_d YZ$.

Problem 6: Let $\mathcal{F}' = \sigma(X^{-1}(\mathcal{A}))$. Since $X^{-1}(\mathcal{A}) \subseteq \sigma(X)$, $\mathcal{F}' \subseteq \sigma(X)$. For the reverse inclusion, let $\mathcal{E} = \{E \in \mathcal{G} : X^{-1}(E) \in \mathcal{F}'\}$. $\mathcal{E}$ is a σ-algebra because the mapping $E \mapsto X^{-1}(E)$

preserves complements and countable unions, and $\mathcal{F}'$ is a σ-algebra. Since $\mathcal{A} \subseteq \mathcal{E}$, $\sigma(\mathcal{A}) = \mathcal{G} \subseteq \mathcal{E}$, so $\mathcal{E} = \mathcal{G}$. It follows that $\sigma(X) = \{X^{-1}(E) : E \in \mathcal{G}\} \subseteq \mathcal{F}'$, so $X^{-1}(\mathcal{A})$ generates $\sigma(X)$. Note that the reasoning recapitulated the proof of Theorem 4.1 in the lecture notes.