Grading: 1, 2, 3, 4, 5 (each 8 pts).

Problem 1: **a)** Let $\mu = E[X_1]$. Since neither σ^2 nor $\overline{V}_n$ change when each X_i is replaced by $X_i - \mu$, we may assume that $\mu = 0$. Then each $E[X_i^2] = \sigma^2$ and $E[X_i X_j] = 0$ for $i \neq j$. It follows that $E[(X_k - \overline{X}_n)^2] = \frac{(n-1)^2}{n^2} E[X_k^2] + \sum_{i \neq k} \frac{1}{n^2} E[X_i^2] = (\frac{(n-1)^2}{n^2} + \frac{n-1}{n^2}) E[X_1^2] = \frac{n-1}{n}\sigma^2$ since all the cross terms have expectation zero. Finally, $E[\overline{V}_n] = \frac{1}{n-1}\sum_{k=1}^n E[(X_k - \overline{X}_n)^2] = \frac{n}{n-1} \cdot \frac{n-1}{n}\sigma^2 = \sigma^2$. **b)** Again assume $\mu = 0$. Since $E[X_1^2] = \sigma^2 < \infty$, the classical Weak LLN (Theorem 2.2.9) implies that $\frac{1}{n}\sum_{k=1}^n X_k^2 \to_p \sigma^2$. Also, $\frac{1}{n}\sum_{k=1}^n [(X_k - \overline{X}_n)^2 - X_k^2] = \frac{1}{n}\sum_{k=1}^n [-2X_k \overline{X}_n + \overline{X}_n^2] = \frac{1}{n}(-2\overline{X}_n \sum_{k=1}^n (X_k) + n\overline{X}_n^2) = -\overline{X}_n^2$. Since $\overline{X}_n \to_p 0$ again by the Weak LLN, Homework 5 Problem 2 implies that $\overline{X}_n \cdot \overline{X}_n \to_p 0 \cdot 0$. Add this to the previous result that $\frac{1}{n}\sum_{k=1}^n X_k^2 \to_p \sigma^2$. The conclusion is that $\frac{1}{n}\sum_{k=1}^n (X_k - \overline{X}_n)^2 \to_p \sigma^2 + 0$. We can replace the initial fraction $\frac{1}{n}$ with $\frac{1}{n-1}$ since $\frac{n}{n-1} \to 1$. Thus $\overline{V}_n \to_p \sigma^2$.

Problem 2: **a)** Using the triangle inequality, $||Z_1||_p \leq ||X_1 \mathbf{1}_{|X_1|>C}||_p + ||E[X_1 \mathbf{1}_{|X_1|>C}]||_p$. The latter term is a constant so it equals $|E[X_1 \mathbf{1}_{|X_1|>C}]| \leq E|X_1 \mathbf{1}_{|X_1|>C}| = ||X_1 \mathbf{1}_{|X_1|>C}||_1$. After, since $|X_1|^p \mathbf{1}_{|X_1| \leq C} \leq |X_1|^p$ and $||X_1||_p < \infty$, by the DCT $\lim_{n\to\infty} E[|X_1|^p \mathbf{1}_{|X_1| \leq C}] = E[|X_1|^p]$ from where we can get $||X_1 \mathbf{1}_{|X_1 > C|}||_p \to 0$ so is less than $\delta/2$ for a sufficiently large C. Writing a similar argument for $||X_1 \mathbf{1}_{|X_1|>C}||_1$ and combining we get the desired result. **b)** Let us pick C large enough so $||Z_1||_p < \delta/2$, then using the triangle inequality we have that $||S_n||_p \leq ||W_1 + \cdots + W_n||_p + ||Z_1||_p + \cdots + ||Z_n||_p \leq ||W_1 + \cdots + W_n||_p + n||Z_1||_p$ from where $||S_n/n||_p \leq ||W_1 + \cdots + W_n||_2/n + ||Z_1||_p \leq (\sum_{i=1}^n E[W_i^2])^{0.5}/n + \delta/2 \leq 2C/\sqrt{n} + \delta/2 < \delta$ for $n > [(4C/\delta)^2]$. Hence, $||S_n/n||_p \to 0$, i.e. $S_n/n \to 0$ in L^p.

Problem 3: **a)** Since $U_1, U_2, ...$ are i.i.d and f is measurable then by the Theorem 2.1.6 $f(U_i)$ are independent. They are identically distributed because $P(f(U_i) \in A) = P(U_i \in f^{-1}(A)) = m(f^{-1}(A))$ is the same for each i (m is the Lebesgue measure on $[0,1]$). Each $E|f(U_i)| = \int_{[0,1]} |f(y)| dm(y) = \int_0^1 |f(y)| dy < \infty$. Hence, we can use the Theorem 2.2.9 to get $\sum_{i=1}^n f(U_i)/n \to_p E[f(U_1)] = \int_0^1 f(y) dy = I$, as desired. **b)** By Chebyshev's inequality we have: $P(|I_n - I| > an^{-0.5}) \leq (n/a^2) \operatorname{Var} I_n = (n/a^2) \sum_{i=1}^n \operatorname{Var} f(U_i)/n^2$, by Theorem 2.2.1, since $\int_0^1 f(x)^2 dx < \infty$ i.e. $E(f(U_i))^2 < \infty$ so $\operatorname{Var} f(U_i) = E(f(U_i)^2) - (Ef(U_i))^2 = \int_0^1 f(x)^2 dx - (\int_0^1 f(x) dx)^2 =: J$. Thus, $P(|I_n - I| > an^{-0.5}) \leq (n/a^2) Jn/n^2 = J/a^2$.

Problem 4: Using the integral test for convergence and since $E[|X_i|] = \sum_{k=2}^\infty C/(k \log k)$ we get that $E[|X_i|] = \infty$. Next, note that $nP(|X_i| > n) = n \sum_{j=n+1}^\infty C/(j^2 \log j) \leq \frac{Cn}{\log n} \sum_{j=n+1}^\infty \frac{1}{j(j-1)} = \frac{Cn}{\log n} \sum_{j=n+1}^\infty (\frac{1}{j-1} - \frac{1}{j}) = C/\log n$, so $\lim_{n\to\infty} nP(|X_i| > n) = 0$ and by the Weak LLN (Theorem 2.2.7) $S_n/n - \mu_n \to 0$ for $\mu_n = E[X_1 \mathbf{1}_{|X_1| \leq n}] = \sum_{j=2}^n (-1)^j C/(j \log j)$. Note that the latter sum is convergent as a monotonic alternating series whose terms converge to 0 and denote the number it converges to by μ. Hence, $S_n/n \to_p \mu$, which completes the proof.

Problem 5: Let $X_{n,k} = X_k$ and $a_n = nE[X_1\mathbf{1}\{X_1 \le b_n\}]$. Theorem 2.2.6 says that under certain conditions $(S_n - a_n)/b_n \to_p 0$. We will show that (1) $b_n \le n/\log_2 n$ for n sufficiently large; (2) $a_n/(n/\log_2 n) \to -1$; and (3,4) the two conditions of Theorem 2.2.6 are met. Given these results, $(S_n - a_n)/b_n \to_p 0 \Rightarrow (S_n - a_n)/(n/\log_2 n) \to_p 0 \Rightarrow (S_n - a_n)/(n/\log_2 n) + (a_n + n/\log_2 n)/(n/\log_2 n) \to_p 0 \Rightarrow S_n/(n/\log_2 n) \to_p -1$, as desired. To show (1): Write $m = m(n)$. We have $2^{m-1}(m-1)^{3/2} < n \le 2^m m^{3/2}$ so $(m-1) + \frac{3}{2}\log_2(m-1) < \log_2 n \le m + \frac{3}{2}\log_2 m$. Dividing by m, we see that $(\log_2 n)/m \to 1$ as $n \to \infty$. Hence $b_n = 2^m < 2n/(m-1)^{3/2} \le n/\log_2 n$ for large enough n. (In fact b_n is asymptotically proportional to $n/(\log n)^{3/2}$.) To show (2): Since $(\log_2 n)/m \to 1$ it is enough to show that $ma_n/n \to -1$. We have $a_n/n = E[X_1\mathbf{1}\{X_1 \le b_n\}] = -p_0 + \sum_{k=1}^m p_k(2^k - 1) = -1 + \sum_{k=1}^m 2^k p_k + \sum_{k=m+1}^\infty p_k = -1 + [1 - 1/(m+1)] + \sum_{k=m+1}^\infty p_k$ where the last equality used the telescoping sum. Thus $ma_n/n = -m/(m+1) + m\sum_{k=m+1}^\infty p_k$. The first term converges to -1. The second term is $m\sum_{k=m+1}^\infty 1/[2^k k(k+1)] \le (1/m)\sum_{k=m+1}^\infty 1/2^k = 1/(m2^m) \to 0$. To show (3): We must check that $nP(X_1 > b_n) \to 0$. This quantity is $n\sum_{k=m+1}^\infty 1/[2^k k(k+1)] \le (n/m^2)\sum_{k=m+1}^\infty 1/2^k = n/(m^2 2^m) \le m^{-1/2}$ since $n/2^m \le m^{3/2}$. To show (4): We must check that $(1/b_n^2)nE[X_1^2\mathbf{1}\{X_1 \le b_n\}] \to 0$. This quantity is $(n/2^{2m})[p_0 + \sum_{k=1}^m p_k(2^k-1)^2] \le (m^{3/2}/2^m)[1 + \sum_{k=1}^m 2^k/k(k+1)]$. The term $(m^{3/2}/2^m)\cdot 1$ tends to 0. Also, if $j = \lfloor m - 2\log_2 m \rfloor$ then $\sum_{k=1}^j 2^k/k(k+1) + \sum_{k=j+1}^m 2^k/k(k+1) \le 2^j\sum_{k=1}^j 1/k(k+1) + (j+1)^{-2}\sum_{k=j+1}^m 2^k$. Using the telescoping sum, the first term is less than $2^j \cdot 1 \le 2^{m-2\log_2 m} = 2^m/m^2$. The second term is less than $(m - 2\log_2 m)^{-2}2^{m+1}$. When multiplying by $m^{3/2}/2^m$ we obtain an upper bound of $m^{-1/2} + 2m^{3/2}(m - 2\log_2 m)^{-2}$, which tends to 0. We have now shown properties (1)-(4), so the proof is complete.