THREE DIMENSIONAL MANIFOLDS, SPRING 2016

1. Lecture 1

Some topics I'd like to cover this semester:

(1) Sphere (Kneser) and Torus (JSJ) decompositions of 3-manifolds.
(2) The Sphere and Loop theorems of Papakyriakopoulos.
(3) Geometric structures on 3-manifolds.
(4) Specifically, hyperbolic geometry.
(5) Sutured manifolds and Agol's Fibering criterion.

Some references: For the first two items, Hatcher's *Notes on Basic 3-Manifold Topology* (`https://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html`) are a good source, see also books by Hempel, Jaco, Schultens [1, 2, 6]. For geometric structures in general, see Thurston's *Three-Dimensional Geometry and Topology* [8]. For the structures which come up in the geometrization theorem, see Scott's article `http://blms.oxfordjournals.org/content/15/5/401.full.pdf+html` [7]. For more on hyperbolic geometry, see Ratcliffe's *Foundations of hyperbolic manifolds* [4]. For the last item, see Scharlemann [5].

1.1. Examples of 3-manifolds.

An n–manifold is a Hausdorff, second countable space, which is locally homeomorphic to $\mathbb{R}^n$. Examples of 3–manifolds are euclidean space $\mathbb{R}^3$, the 3–torus $T^3 = S^1 \times S^1 \times S^1$, the 3-sphere S^3, and open subsets of these, for example knot complements.

A surprisingly general way to build a 3–manifold is to start with a surface Σ and a homeomorphism $\phi\colon \Sigma \to \Sigma$, and form the *mapping torus* $M_\phi = \Sigma \times [0,1]/(x,0) \sim (\phi(x),1)$. (In other words, glue the boundary components of $\Sigma \times [0,1]$ to each other using the homeomorphism ϕ.) Such a mapping torus is also called a *surface bundle over a circle.*

1.2. $(\mathcal{G}, X)$ structures.

(cf. [8, Chapter 3]) Let X be a topological space. A *pseudogroup* $\mathcal{G}$ on X is a set of homeomorphisms between open subsets of X, so that

(1) The domains of $f \in \mathcal{G}$ cover X;
(2) $\mathcal{G}$ is closed under restriction, composition, and inverse; and
(3) (locality) If $f\colon U \to V$ is a homeomorphism, and U has an open cover $\{U_\alpha\}$ so that every restriction $f_\alpha = f|U_\alpha$ is in $\mathcal{G}$, then f is in $\mathcal{G}$.

Any space X admits at least two pseudogroups:

- The trivial pseudogroup $\mathcal{G} = \{1_U \mid U \text{ open in } X\}$, and
- the maximal pseudogroup $\mathsf{TOP} = \{f\colon U \to V \mid U, V \text{ open}, f \text{ a homeo}\}$.

Most interesting pseudogroups lie in between.

If X is a space, an $\mathcal{G}$ a pseudogroup on that space, we can define a $(\mathcal{G}, X)$*–space* to be a space M, together with an open covering by *charts* $U_\alpha \subseteq M$ and chart maps $\phi_\alpha\colon U_\alpha \to X$, so that the *transition maps* $\gamma_{\alpha\beta}$ lie in $\mathcal{G}$. For an arbitrary pair α, β, let $U_{\alpha\beta} = U_\alpha \cap U_\beta$; the transition map $\gamma_{\alpha\beta} \colon \phi_\alpha(U_{\alpha\beta}) \to \phi_\beta(U_{\alpha\beta})$ is the composition $\gamma_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$.

When X and M are manifolds, we call M a $(\mathcal{G}, X)$*–manifold*, or just $\mathcal{G}$–manifold if X is understood.

Examples of $(\mathcal{G}, X)$–structures (where $X = \mathbb{R}^n$ if it is not mentioned):

- A TOP–manifold is just a manifold.
- DIFF is the pseudogroup generated by all C^∞ diffeomorphisms of $\mathbb{R}^n$. A DIFF–manifold is the same thing as a smooth manifold.
- PL is generated by restrictions to open subsets of *piecewise linear* homeomorphisms between n–dimensional polyhedra in $\mathbb{R}^n$.
- If X is a Riemannian manifold, and $\mathcal{G}$ is generated by $\mathrm{Isom}(X)$, we get some kind of metric structure on any $\mathcal{G}$–manifold M. For example if $X = \mathbb{H}^n$, then M is a *hyperbolic* manifold.
- One can also define real analytic, complex, symplectic, contact, etc. structures using this scheme.

Fixing $(\mathcal{G}, X)$, a $\mathcal{G}$*–isomorphism* is a homeomorphism $h \colon M_1 \to M_2$, so that, for any charts $\phi_\alpha \colon U_\alpha \to X$ for M_1 and $\psi_\beta \colon V_\beta \to X$ for M_2, $\psi_\beta \circ h \circ \phi_\alpha^{-1} | \phi_\alpha(h^{-1}(V_\beta))$ is in $\mathcal{G}$.

The pseudogroup PL does not contain DIFF, but there is a sense in which it is intermediate: every DIFF–structure uniquely determines a PL–structure, through the existence of *smooth triangulations* (see Lecture 3).

WARNING: In high dimensions, the classification problems for TOP–manifolds, PL–manifolds, and DIFF–manifolds all differ.

A glimpse of how the categories differ is given in the following table, which addresses the question: Fixing n and $\mathcal{G}$, is every homotopy n–sphere $\mathcal{G}$–isomorphic to the standard sphere S^n?

<table>
<tr><th>dim</th><th>TOP</th><th>PL</th><th>DIFF</th></tr>
<tr><td>≤ 2</td><td colspan="3">TRUE (classical)</td></tr>
<tr><td>3</td><td colspan="3">TRUE (Perelman*)</td></tr>
<tr><td>4</td><td>TRUE (Freedman*)</td><td colspan="2">OPEN</td></tr>
<tr><td>5, 6</td><td colspan="2" rowspan="2">TRUE (Smale*, Stallings, Newman)</td><td>TRUE (Milnor–Kervaire)</td></tr>
<tr><td>≥ 7</td><td>FALSE (Milnor*)</td></tr>
</table>

(Note: * after a name indicates not only a Fields medal, but a Fields medal whose citation includes the result in the box!)

In fact, things are much more complicated and interesting even than this table suggests. But in terms of the material in this class we have the following:

Important fact. *In dimensions ≤ 3, the distinctions between* TOP*,* PL*, and* DIFF *"disappear":*

- *Let M be an n–manifold for $n \leq 3$. Then M supports a unique* PL *structure, and a unique* DIFF *structure.*
- *Every* PL *embedding between manifolds of dimension ≤ 3 is isotopic to a smooth one, and vice versa.*

(See Moise's book [3] for more on the above.)

WARNING: Topological "embeddings" can be wild. Examples include wild knots and Alexander's horned sphere. We therefore generally restrict attention to *locally flat* (eg PL or smooth embeddings).

Convention 1.1. *Unless otherwise noted, all maps and manifolds are to be assumed smooth (though corners are ok – see below).*

1.3. **Three-manifolds as we know them today.** Much of the following terminology will be defined later.

Let M be compact closed and oriented:

- M has a unique decomposition into *prime* summands (Kneser decomposition).
- If M is prime, it decomposes canonically along tori into pieces which are either *Seifert fibered* or *atoroidal* (JSJ decomposition).

The above is 20th century knowledge. But in the 21st century, we know vastly more:

- Thurston's *Geometrization Conjecture* is true. The pieces from the JSJ decomposition support geometric structures ($(\mathcal{G}, X)$–structures where $\mathcal{G}$ is generated by a Lie group acting transitively by isometries on X) of eight specific types, including hyperbolic, spherical, and euclidean.
- A special case of the last statement is the Poincaré Conjecture in dimension 3.
- Hyperbolic manifolds are very well understood, thanks to the proofs of the Ending Lamination Conjecture and the Marden Conjecture, which are probably beyond the scope of this class.
- Hyperbolic and many other manifolds are finitely covered by surface bundles over S^1 (they *virtually fiber*).

The last item uses a combination of geometric group theory techniques with *Agol's fibering criterion*. It is a goal of this course to do enough 3–manifold topology to understand a proof of Agol's fibering criterion.

References

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[2] W. Jaco. *Lectures on three-manifold topology*, volume 43 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1980.

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