THREE-MANIFOLDS NOTES

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Last Time. We were in the middle of proving that if M is a compact irreducible SFS and $\Sigma \subseteq M$ is essential, then Σ is isotopic to either a vertical or horizontal surface. We showed this theorem modulo the claim that Σ (chosen with minimal complexity) was incompressible or a union of meridian disks in M_1. (i.e. $\Sigma_1 \subseteq M_1$ was incompressible.) We left with the case that $\partial\Sigma_1$ was vertical

Lemma 1. *If Σ is incompressible with vertical or horizontal boundary, then Σ_1 is planar.*

Σ_1 is Planar. Suppose $\partial\Sigma_1$ is vertical. Then there are no type 3 boundary components. If there is a compressing disk in M_1, then its boundary must bound a subsurface of Σ_1 all of whose boundary components are type 1, and it has at least one such. Let F be a disk in Σ with the same boundary. Use the resulting 3-ball to isotope Σ to have lower complexity.

This completes the proof of the theorem.

Next. Let Σ be a 2-sided surface in an irreducible Seifert-Fibred Surface M.

- If Σ is horizontal, then it is essential.
- If Σ is vertical, then it is an essential surface, a torus cutting off a model fibred solid torus, or a boundary-parallel annulus cutting off a model fibred solid torus.

ORBIFOLDS

For reference, see Scott, "The Geometry of 3-Manifolds".

Let X be a Hausdorff, second-countable space. For each point $x \in X$ and each open neighborhood U of x, an *orbifold chart* consists of an open neighborhood $U_\alpha \subseteq U$ of x, an open set $\tilde{U}_\alpha \subseteq R^n$, a finite group Γ_x of diffeomorphisms of R^n fixing the origin, and a commutative diagram

$$\begin{array}{ccc} \tilde{U}_\alpha & \longrightarrow & \tilde{U}_\alpha/\Gamma_x \\ & {\scriptstyle\tilde{\varphi}_\alpha}\searrow \quad \swarrow{\scriptstyle\varphi_\alpha} & \\ & U_\alpha & \end{array}$$

such that φ_α is a homeomorphism, and $\tilde{\varphi}_\alpha(0) = x$.

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A *smooth orbifold atlas* on X is a collection containing an orbifold chart for each $x \in X$ and open neighborhood U of x, subject to the following compatibility condition: for any $z \in U_\alpha$ and any orbifold chart $U_\beta \subseteq U_\alpha$ with $\tilde{\varphi}_\beta(0) = z$, there is a monomorphism $h_{\alpha\beta} : \Gamma_z \to \Gamma_x$ and a $h_{\alpha\beta}$-equivalent embedding $\psi_{\alpha\beta} : \tilde{U}_\beta \to \tilde{U}_\alpha$ such that

$$\begin{array}{ccc} \tilde{U}_\beta & \xrightarrow{\psi_{\alpha\beta}} & \tilde{U}_\alpha \\ \downarrow{\scriptstyle \tilde{\varphi_\beta}} & & \downarrow{\scriptstyle \tilde{\varphi}_\alpha} \\ U_\beta & \hookrightarrow & U_\alpha \end{array}$$

commutes

Remark. In particular, Γ_x doesn't depend on α.

A space X together with a maximal smooth orbifold atlas is a *smooth orbifold*. If Γ_x and $\psi_{\alpha\beta}$ all lie in a particular pseudogroup $\mathcal{G}$, then we say that X is a $\mathcal{G}$-orbifold.

Remark. We can always take Γ_x to be conjugate to a finite subgroup of $O(n)$ in the smooth setting.

Example 2 (1-Dimensional Orbifolds)**.** The local groups Γ_x are either trivial or $Z/2Z$, so we have four 1-dimensional orbifolds: the real line R, the circle S^1, a ray with a "mirror" at the end, and an interval with two mirrors. See figure 1.

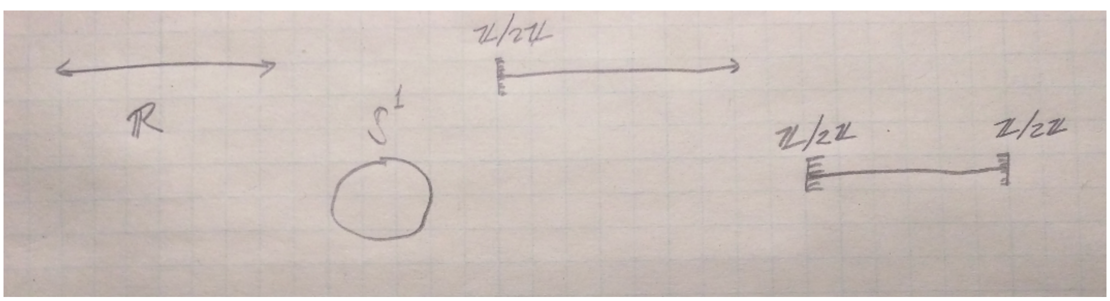

FIGURE 1. All One Dimensional Orbifolds

Example 3 (2-Dimensional Orbifolds)**.** We can have four kinds of local groups: the trivial group, Z/nZ acting as a rotation (called a "gyration point of order n"), $Z/2Z$ acting as a reflection, and D_n (called a "corner reflector of order n"). See figure 2.

Given a Seifert-fibred space M, the *decomposition space* B given by collapsing the fibres has the structure of an orbifold.

Example 4 (Base Spaces for Model Fibred Solid Torus & Klein Bottle)**.** The base space of a model fibred solid torus is a disk with one gyration point. The base space of a Klein bottle is a half disk where a portion of the boundary is a mirror. See figure 3

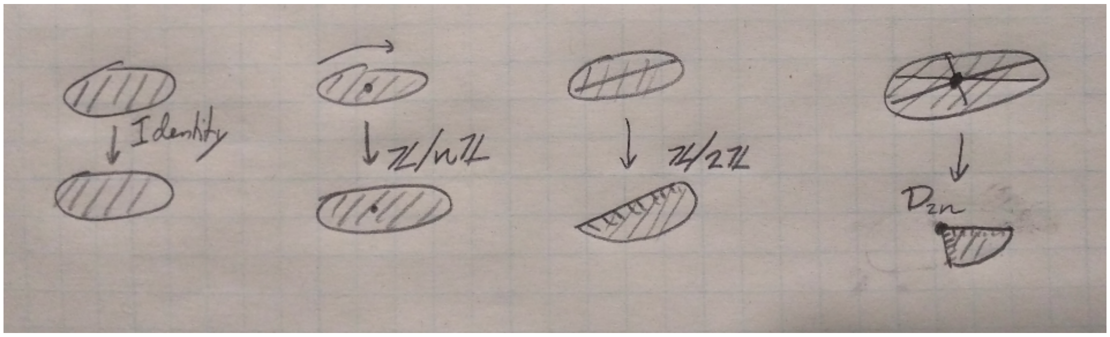

FIGURE 2. The Possible Local Pictures For Two Dimensional Orbifolds

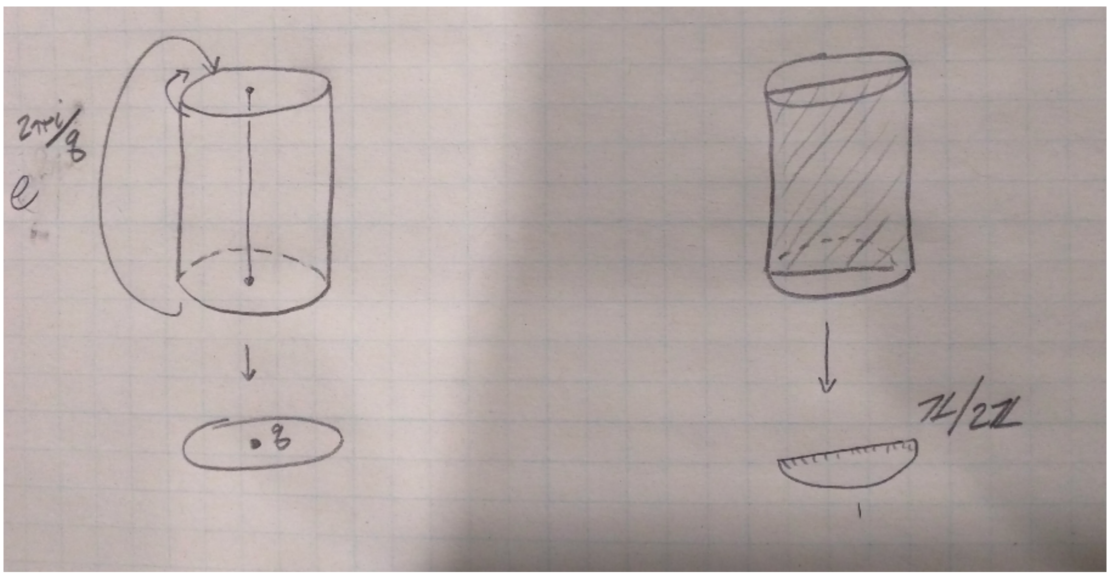

FIGURE 3. Base Spaces of a Model Fibred Solid Torus and a Model Fibred Klein Bottle

COVERING SPACES OF ORBIFOLDS

A *covering orbifold* is a continuous map $\pi : Y \to X$ between orbifolds such that for each $x \in X$ there is an orbifold chart U_α around x such that $\pi^{-1}(U_\alpha) = \bigsqcup_i V_i$ is a disjoint union of orbifold charts around the points of $\pi^{-1}(x)$ with an injection $\Gamma_i \to \Gamma_x$ and an equivariant homeomorphism $\tilde{V}_i \to \tilde{U}_\alpha$ making the diagram

$$\begin{array}{ccc} \tilde{V}_i & \xrightarrow{\cong} & \tilde{U}_\alpha \\ \downarrow & & \downarrow \\ V_i & \xrightarrow{\pi} & U_\alpha \end{array}$$

Example 5 (Coverings of Orbifolds)**.** Figure 4 shows some examples of coverings of a 1-dimensional orbifold.

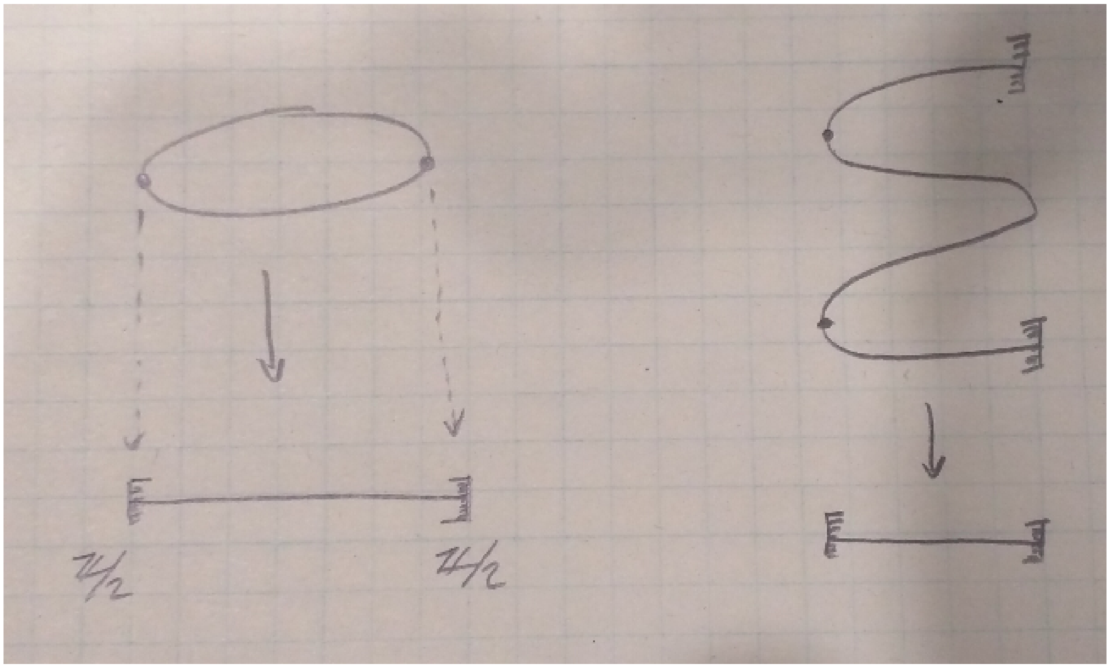

FIGURE 4. Examples of Coverings of Orbifolds

The *degree* of a covering π is given by

$$\sum_{y \in \pi^{-1}(x)} [\Gamma_x : \Gamma_y]$$

A *deck transformation* of a covering $\pi : Y \to X$ of orbifolds is an orbifold isomorphism $Y \to Y$ that commutes with the covering map. A *regular cover* is a cover such that for all $\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)$, there is a deck transformation taking $\tilde{x}_1$ to $\tilde{x}_2$.

Remark. It is a fact that every orbifold has a universal cover that is unique up to isomorphism.

Definition 6. The orbifold fundamental group $\pi_1^{orb}(X)$ of an orbifold X is the deck group of the universal orbifold cover of X

Example 7. Let X be the orbifold with two gyration points, one of order 2 and another of order 3. Then $\pi_1^{orb}(X) \cong PSL(2, Z) \cong Z/2Z * Z/3Z$.

Definition 8. An orbifold is *good* if its universal cover is a manifold. It is *very good* if there is a finite degree cover by a manifold.

Example 9. All 1-orbifolds are very good.

Example 10. Flipping a torus end over end yeilds a sphere with four gyration points of order 2. See figure 5.

Example 11 (Bad 2-Orbifolds)**.** We have examples of orientable as well as non-orientable bad 2-orbifolds. See figure 6.

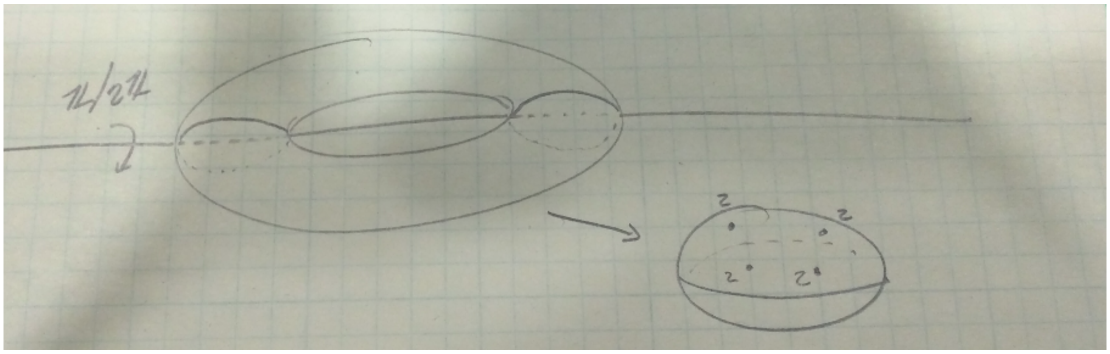

FIGURE 5. Covering an Sphere with Four Gyration Points of Order 2

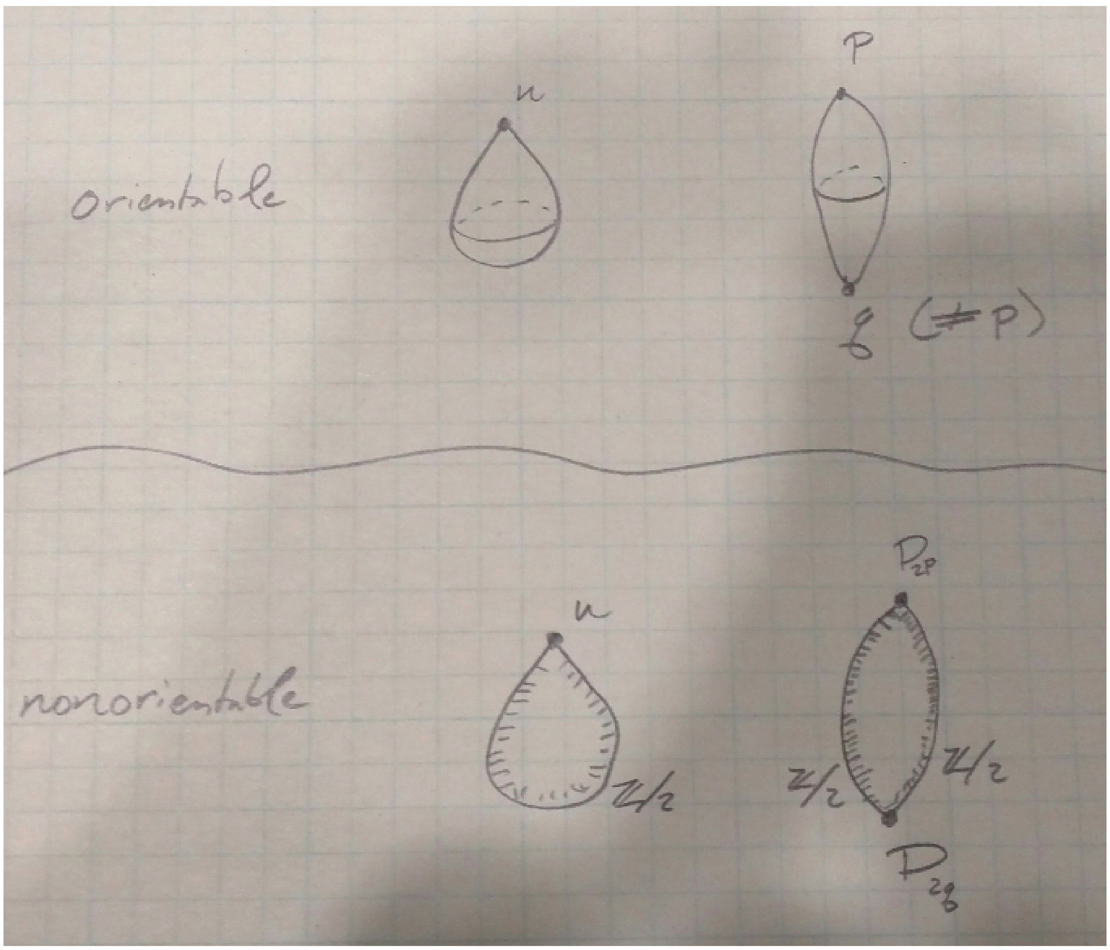

FIGURE 6. Examples of Orientable and Nonorientable Bad 2-Orbifolds

Exercise 1. Realize the bad orientable 2-orbifolds as bases of Seifert fibrings