# THREE-MANIFOLDS NOTES

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**Last Time.** We saw that if $B$ is an orbifold, then there is a *universal orbifold convering space* $\tilde{B} \to B$. The *orbifold fundamental group* $B$, denoted $\pi_1^{\mathrm{orb}} B$, is defined to be the deck group of the universal cover.

As in ordinary covering space theory, there is a "Galois correspondence"

$$\{\text{subgroups of } \pi_1^{\mathrm{orb}} B\}/(\text{conjugacy}) \leftrightarrow \{\text{orbifold covers } \tilde{B} \to B\}/(\text{isotopy}).$$

**Orbifold Euler Characteristic.** We want to define an Euler characteristic $\chi$ of orbifolds so that $\chi$ is multiplicative under finite orbifold covering maps. For instance, we'll have

$$\chi^{\mathrm{orb}}(\text{disk with order-}n\text{ cone point}) = \frac{1}{n},$$
$$\chi^{\mathrm{orb}}(\text{disk with half of its boundary mirrored}) = \frac{1}{2}.$$

We can extend this to a definition of Euler characteristic for a general orbifold by formally dividing the orbifold into pieces marked by a single group $G$, and defining

$$\chi^{\mathrm{orb}}(\text{piece}) = \frac{\chi(\text{piece})}{|G|}.$$

This will satisfy

$$\chi^{\mathrm{orb}}(B) = \frac{1}{n}\chi^{\mathrm{orb}}(\tilde{B}) \qquad \text{for an } n\text{-fold cover } \tilde{B} \to B,$$
$$\chi^{\mathrm{orb}}(A \cup B) = \chi^{\mathrm{orb}}(A) + \chi^{\mathrm{orb}}(B) - \chi^{\mathrm{orb}}(A \cap B).$$

**Example 1.** The following is a complete list of the orientable 2-orbifolds with positive Euler characteristic:

- the "teardrop," a 2-sphere with a single order-$p$ cone point
- the "spindle," a 2-sphere with cone points of orders $p$ and $q$ with $\gcd(p, q) = 1$
- the 2-sphere
- the 2-disk

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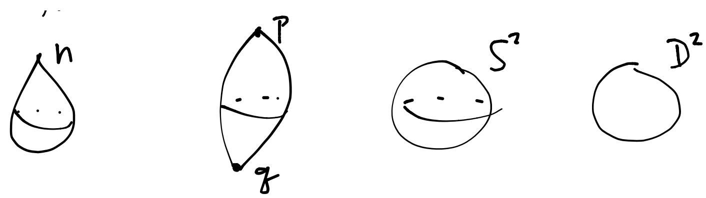



FIGURE 1. Orientable 2-orbifolds with positive Euler characteristic.

**Remark.** Later on we'll use the fact that these are the 2-orbifolds covered by a disk:

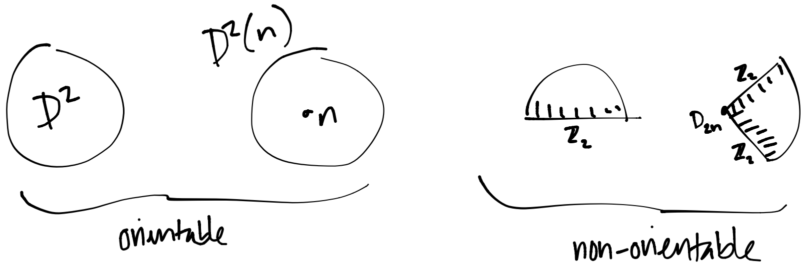



FIGURE 2. The 2-orbifolds covered by $D^2$.

## (BACK TO) ESSENTIAL SURFACES IN SEIFERT FIBERED SPACES

**Lemma 2.** *Let* $M$ *be a SFS*[1], *with* $\pi : M \to B$ *the projection onto the base orbifold. If* $p : \hat{B} \to B$ *is an orbifold covering map, then* $p$ *induces a covering map* $\hat{p} : \hat{M} \to M$. *The cover* $\hat{p}$ *is regular whenever* $p$ *is.*

> **Exercise 1.** Prove Lemma **??**.

**Corollary 3.** *The induced map* $\pi_1 M \to \pi_1^{orb} B$ *is a surjection, with kernel generated by a generic fiber of* $M$.

**Proposition 4.** *Let* $\Sigma \subset M$ *be a 2-sided surface in an irreducible SFS* $M$. *If* $\Sigma$ *is horizontal, then it is essential.*

*Proof.* Let $B$ be the base orbifold of $M$ with projection $\pi : M \to B$. Then $\pi|_\Sigma : \Sigma \to B$ is an orbifold cover, so $\pi_1 \Sigma \to \pi_1^{\text{orb}} B$ is injective. We have the following commutative diagram.

[1] Seifert fibered space; we'll use this abbreviation a lot.

$$\begin{array}{ccc} \pi_1\Sigma & \longrightarrow & \pi_1 M \\ & \searrow & \downarrow \\ & & \pi_1^{orb} B \end{array}$$

Since the diagonal arrow is an injection, the top arrow must be as well. Hence $\Sigma$ is $\pi_1$-injective and therefore incompressible.

Now, we had a lemma a few lectures ago that stated that if $\Sigma$ is inessential but incompressible, then $\Sigma$ is a boundary-parallel annulus. The following exercise completes the proof. □

> **Exercise 2.** Show that a SFS $M$ contains no horizontal boundary-parallel annuli.

**Proposition 5.** *Let $\Sigma \subset M$ be a 2-sided surfaces in a SFS $M$. If $\Sigma$ is vertical, then it is either*

(1) *essential,*

(2) *a vertical torus cutting off a model fibered solid torus,*

(3) *a vertical Klein bottle cutting off a model fibered solid Klein bottle,*

(4) *a vertical annulus cutting of a product solid torus, or*

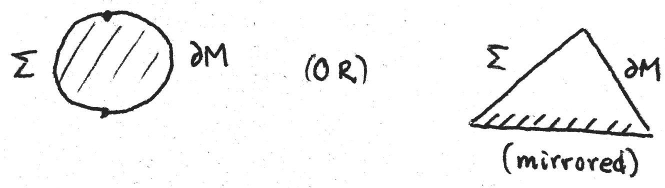
(5) *a vertical Möbius band cutting off a model fibered solid Klein bottle.*

*Proof.* Suppose $\Sigma$ is not essential.

CASE 1: $\Sigma$ is compressible. Let $D$ be a compressing disk for $\Sigma$, so that $D$ is essential in $M\backslash\Sigma$. Since $\chi(D) \neq 0$, $D$ cannot be made to be vertical, so we must be able to make it horizontal. If $M_1$ is the component of $M\backslash\Sigma$ containing $D$ and $B_1$ is the base orbifold of $M_1$, we get an orbifold cover $D \to B_1$. Since $B_1$ contains no corner reflectors, from Figure **??** we see that $B$ is either a disk, a disk with a cone point, or a half-disk with part of its boundary mirrored. In the first two cases it follows that $M\backslash\Sigma$ is a model fibered solid torus, and in the last case it follows that $M\backslash\Sigma$ is a model fibered solid Klein bottle.

CASE 2: $\Sigma$ is incompressible but $\partial$-compressible. As before, a $\partial$-compressing disk $D$ can be made horizontal in some component $M_1$ of $M\backslash\Sigma$, and we get an orbifold covering $D \to B_1$ of the base orbifold $B_1$ of $M_1$. Hence $B_1$ is a disk; part of $\partial B_1$ contains the projection of points in $\partial\Sigma$, and another part contains the projection of points in $\partial M$.

Since the part of $\partial B_1$ coming from points in $\Sigma$ must be connected, it follows that the only possibilities for $B_1$ are as shown below.




In the left case, $M \backslash \Sigma$ is a solid torus (with trivial fibering) and $\Sigma$ is an annulus. In the right case, $M \backslash \Sigma$ is a solid Klein bottle and $\Sigma$ is a Möbius band. This completes the proof. □

## Uniqueness of Minimal Torus Decompositions for SFSs

For the following, let $M$ be an irreducible, orientable, compact SFS.

**Lemma 6** (Lemma A). *If $A$ is an essential annulus in $M$, then $A$ is isotopic to a vertical surface unless $M$ is one of the following exceptional SFSs:*

- $T \times I$, *where* $T = S^1 \times S^1$,
- $K \times I$, *where* $K$ *is a Klein bottle,*
- $T \tilde{\times} I$, *or*
- $K \tilde{\times} I$.

*In the exceptional cases, the fibering on $M$ can be modified so that $A$ is vertical.*

**Lemma 7** (Lemma B). *Any two Seifert fiberings on $M$ agree on $\partial M$, unless $M$ is one of the four exceptional spaces listed above.*

**Lemma 8** (Lemma C). *Assume $N$ is a connected, compact, orientable, irreducible, atoroidal 3-manifold that contains an essential annulus. Then $N$ is a SFS.*

*Proof of Lemma A.* We assume that $A$ cannot be isotoped to be vertical; it follows that $A$ can be isotoped to be horizontal. Let $M_1 = M \backslash A$.

Case 1: $M_1$ is connected. It follows that $M_1 \cong A \times I$ is a trivial I-bundle over $A$. To recover $M$, we glue $A \times 0$ to $A \times 1$ by some automorphism $\varphi$ of $A$. Writing $A = S^1 \times I$, $\varphi : S^1 \times I \to S^1 \times I$ is determined up to isotopy by whether or not it reverses orientation on each factor, yielding four possibilities that correspond to the four exceptional SFSs. In any case, we can fiber $A$ by circles and extend that fibering to all of $M$ so that $A$ is vertical.

Case 2: $M_1$ contains two components, $M_1^+$ and $M_1^-$. Each component is a twisted I-bundle over $\tilde{A}$, which is either an annulus or a Möbius band...

*To be continued!*

□