# THREE-MANIFOLDS NOTES

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## UNIQUENESS OF TORUS DECOMPOSITION

Recall the three lemmas from the previous lecture.

**Lemma A.** *Suppose* $M$ *is an irreducible Seifert fibered space.* $A \subset M$ *is an essential annulus. Then* $A$ *can be made vertical by an isotopy if* $M \in \{T^2 \times I, T^2 \tilde{\times} I, K \times I, K \tilde{\times} I\}$. *In any case,* $A$ *can be made vertical in* some *Seifert fibering of* $M$.

*Proof, continued.* Last time we proved most of lemma A, in particular when $A$ is non-separating.

If $A$ is separating, then it separates $M$ into two pieces $M_1$, $M_2$ of the form in Figure 1. Each piece is an I bundle twisted over a Möbius band (gluing by rotation) or annulus (gluing by reflection). By examining the three possibilities, we see

(1) $M$ can be fibered so that $A$ is vertical.
(2) $M \in \{T^2 \tilde{\times} I, K \times I, K \tilde{\times} I\}$. □

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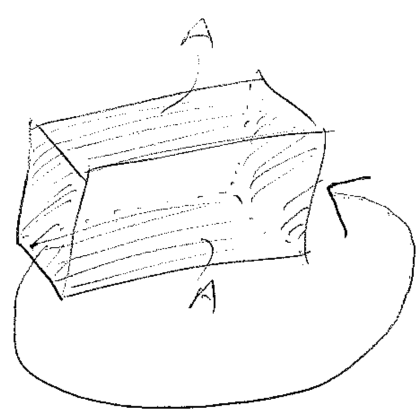


FIGURE 1. Ends of $M_1$ are glued by reflection or rotation.

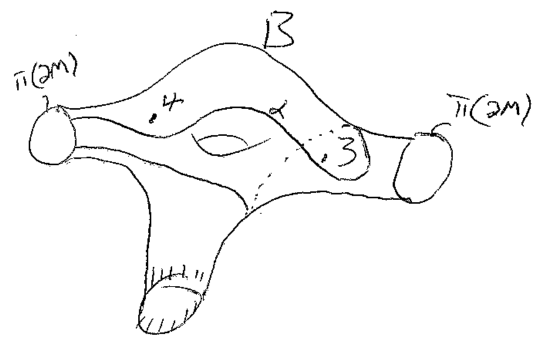

FIGURE 2. A not very simple base orbifold and essential arc.

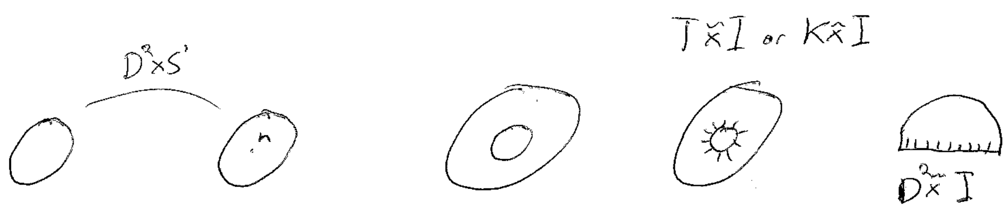

FIGURE 3. All the very simple orbifolds.

**Lemma B.** *Suppose* M *is an irreducible Seifert fibered space and* $M \notin \dagger = \{T^2 \times I, T^2 \tilde{\times} I, K \times I, K \tilde{\times} I, D^2 \times S^1, D^2 \tilde{\times} S^1\}$*. Then any two Seifert fiberings agree on* $\partial M$*.*

*Proof.* **Claim:** For any component T of $\partial M$ there is an essential vertical annulus with at least one boundary component, unless M is in the list $\dagger$.

**Proof of claim:** Let B be the base orbifold of M. Unless B is *very simple*, B contains an essential arc $\alpha$ with $\partial\alpha \cap \pi(T) \neq \emptyset$. B is not *very simple* (for example Figure 2) if any of the following hold:

- B has positive genus.
- B has at least two orbifold features.
- B has at least three boundary components or orbifold features.

The very simple disks are listed in Figure 3 and all either have an essential arc or correspond to $M \in \dagger$.

From the essential arc, $A = \pi^{-1}(\alpha)$ is a vertical annulus. It is in fact essential. The annulus determines the fibering on T up to isotopy. Lemma A implies that

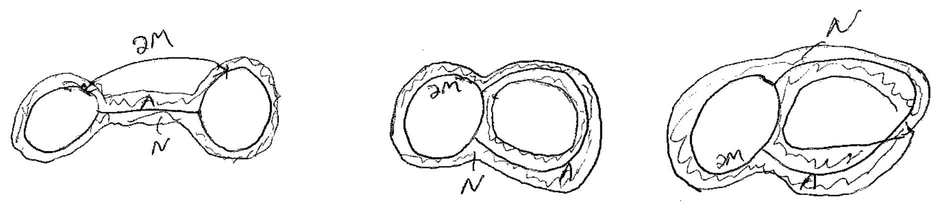


FIGURE 4. Base surfaces from Lemma C.

$A$ can be made vertical with respect to any fibering, and so any two fiberings must agree on $T$. □

**Definition 1.** An arc $\alpha \subset B$ is *essential* if there is no disk $D$ embedded in $B$, with $\partial D + \alpha \cup \beta$ where $\beta$ is an arc of $\partial B$.

**Lemma C.** *Suppose $M$ is compact, connected, orientable, irreducible, atoroidal and contains an essential annulus. Then $M$ is a Seifert fibered space.*

*Proof.* Let $A$ be an essential annulus. Let $N$ be a regular neighborhood of $A \cup T$, where $T$ is the one or two components of $\partial M$, which intersect $A$. Then $\partial N \backslash (\partial M \cap \partial N)$ has one or two tori components (always 2-sided). Since $M$ is atoroidal, $\partial N \backslash (\partial M \cap \partial N)$ is either boundary parallel or bounds a solid torus. If boundary parallel, then $M$ is an $S^1$ bundle over one of the surfaces in Figure 4 and is Seifert fibered.

Otherwise,
**Fact:** Almost any fibering of $\partial(D^2 \times S^1)$ extends to a Seifert fibering of $D^2 \times S^1$. The one exception is a fibering by meridian curves; in that case, $A$ is compressible. (The other cases correspond to Dehn fillings.) □

**Theorem 2** (Torus Decomposition)**.** *Suppose $M$ is orientable, compact, and irreducible. There exists an embedded collection $T$ of incompressible tori cutting $M$ into atoroidal and Seifert fibered pieces. A minimal collection, where removing any torus would fail to cut $M$ into atoroidal and Seifert fibered pieces, is unique up to isotopy.*

*Proof.* Existence follows from a previous theorem bounding the tori needed to cut $M$ into atoroidal pieces.

For uniqueness: suppose $T = T_1 \cup \ldots \cup T_m$ and $T' = T_1' \cup \ldots \cup T_n'$ are minimal. We will induct on $\min\{m, n\}$, supposing that $m \leq n$.

Base case: $m = 0$. If $n > 0$ then $T$ is non-minimal because $T'$ is a (empty) subset of $T$.

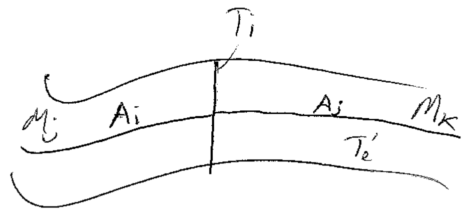

FIGURE 5. Decomposition corresponding to $T_i$.

Next, assume that no $T_i \sim T_j'$, since otherwise we can split along the common torus and apply the induction. We can assume that $T_i \cap T_j'$ is essential in $T_i$ and $T_j'$ by standard arguments of incompressibility and irreducibility. We want to show that after isotopy, $T \cap T' = \emptyset$. Assume that $T$ and $T'$ are chosen in their isotopy classes so that the number of components of $T \cap T'$ is minimal. Suppose that $M$ cut along $T$ is $M_1, \cdots, M_v$, and $M$ cut along $T'$ is $M_1', \cdot, M_w'$.

**Claim:** Any component of $T' \cap M_j$ is an essential annulus in $M_j$.
**Proof:** Otherwise, Lemma 2 from before says any inessential component is boundary parallel. Push that annulus through $T$ to decrease the components of $T \cap T'$.

It follows from Lemma C that any $M_j$ with boundary intersecting $T'$ is Seifert fibered.

Consider a torus $T_i \subset T$ which intersects $T'$ in a component $T_l'$. A curve on $T_l'$ is the boundary of two annuli, $A_j \subset M_j \cap T_l'$ and $A_k \subset M_k \cap T_l'$ (see Figure 5).

**Case 1:** $M_j \neq M_k$. Use Lemma A to refiber $M_j$ and $M_k$ so the fiberings agree on $T_i$. We can then remove $T_i$ and glue $M_j$ and $M_k$ to get a Seifert fibered space, so $T$ was not minimal.

**Case 2:** $M_j = M_k$. If $M_j$ is not one of the "exceptional" manifolds then the fiberings on either side agree up to isotopy from lemma B, so we can again discard $T_i$. Otherwise...

*Finalizing arguments to come next lecture.* □