THREE MANIFOLDS NOTES

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1. Uniqueness of Torus Decomposition

Recall the three lemmas from last lecture, Lemma A, B, and C, (shown in the notes of last class.) We finish the proof of the following theorem.

Theorem 1.1. *Suppose M is orientable, compact, and irreducible. There exists an embedded collection T of incompressible tori cutting M into atoroidal and Seifert fibered pieces. A minimal collection, where removing any torus would fail to cut M into atoroidal and Seifert fibered pieces, is unique up to isotopy.*

Proof. (Continued) **Case 2**: $M_j = M_k$. If M_j is not one of the "exceptional" manifolds then the fiberings on either side agree up to isotopy from lemma B, so we can again discard T_i. Otherwise, the only possible exceptional case (with two torus boundary components) is $T^2 \times I$. Then A_j must be simple closed curves $\times I$. So the two boundaries of A_j are parallel, and M_j can be fibered. Again we can extend the fiberings over T_i, contradicting the minimality.

Thus we may assume $T \cap T' = \emptyset$. (and no T_i is isotopic to any T'_j) Note that if some M_j contains a torus of T'_i, that M_j is Seifert fibered; and vice versa. $T \cup T'$ cuts M into pieces $\{N_p\}$. Consider a torus T_i of T contained in a Seifert fibered M'_j. See Figure 1. We can assume all of M_k, M'_j, M_j are Seifert fibered. Our goal is to show that the fiberings on M_j and M_k agree on T_i. It suffices to show that the fiberings on N_p coming from M_j, M'_j can be made to agree on T_i. If N_p, N_q are not exceptional to † (shown in Lemma B in the previous class notes), any fiberings will agree on the boundary. In this case we conclude fiberings agree on T_i, contradicting the minimality.

The exceptional cases are

(1) $N_p = S^1 \times D^2$ (T_i is incompressible, so this wouldn't happen)
(2) $N_p = T^2 \times I$ (No parallel tori, so this wouldn't happen)
(3) $K\tilde{\times}I$ ($N_p = M_j \subseteq M'_j$; in this case we can refiber M_j to agree with M'_j)

The reasoning for N_q is totally symmetric. And the proof is done. □

Facts to know: More generally a torus knot complement is Seifert fibered.
Extensive reading: Swallow-follow torus Satellite knot. Below is a picture showing the satellite knot $K \subseteq S^3$, Figure 2. (Picture from the website of Hayashi)

2. The Loop and Sphere Theorems

Theorem 2.1 (Loop Theorem)**.** *Let M be a 3 manifold with boundary. Suppose there is a map $f : (D^2, \partial D^2) \to (M, \partial M)$ so that $f|_{\partial D^2}$ is not null-homotopic in ∂M. Then there is an embedding $(D^2, \partial D^2) \to (M \partial M)$ with the same property.*

Proof. Proof in the next lecture. □

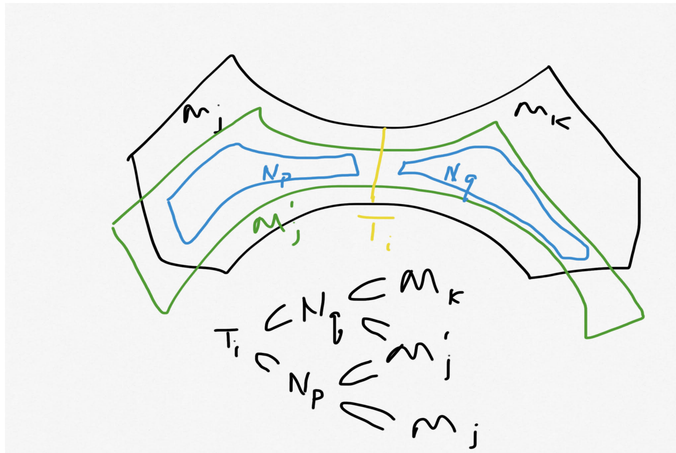

FIGURE 1

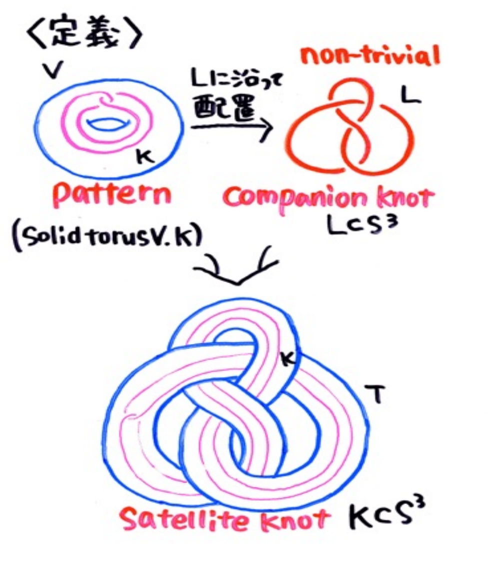

FIGURE 2

Theorem 2.2 (Sphere Theorem)**.** *Let M be a connected 3 manifold with $\pi_2(M) \neq 0$. Then either*

(1) *there is an embedded S^2 in M representing a nontrivial element of $\pi_2(M)$, or*
(2) *there is an embedded 2-sided $\mathbb{R}P^2$ in M representing a nontrivial element of $\pi_2(M)$.*

Proof. Proof in the next lecture. □

Let's take a look at the corollaries of the loop theorem.

Corollary 2.3 (Dehn's Lemma)**.** *If α is an embedded null-homotopic circle in ∂M, where M is any 3 manifold, then α bounds an embedded disk in M.*

Proof. Let N be a regular neighborhood of α in ∂M. Let $M' = M \backslash (\partial M \backslash N)$. α is still null-homotopic in M', and $\partial M' = N$. Apply the look theorem to get embedded disk in M'. The boundary of this disk is isotopic to α. □

Corollary 2.4. *Let $\Sigma \subset M$ be a 2-sided surface where M is a 3 manifold. Σ is incompressible if and only if Σ is π_1-injective.*

Proof. One direction was already shown. □