3-MANIFOLDS

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1. Loop Theorem

Last time, we stated the Loop Theorem and some corollaries.

Theorem (Loop Theorem). *Let M be a 3-manifold with boundary. Given $f : (D^2, \partial D^2) \to (M, \partial M)$ with $f|_{\partial D}$ nontrivial in $\pi_1(\partial M)$, there is an embedding $f' : (D^2, \partial D^2) \to (M, \partial M)$ with the same property.*

Remark. if $f|_{\partial D} \notin N \lhd \pi_1(\partial M, \mathrm{pt})$ then we can ensure the same is true of $f'|_{\partial D}$ (the proof is essentially the same).

Corollary 1.1 ((Dehn's Lemma)). *An embedded curve in ∂M which is null-homotopic in M bounds an embedded disk in M.*

Corollary 1.2 (Corollary 2). *If $\Sigma \subseteq M$ is a 2-sided, properly embedded surface and $\Sigma \neq D^2, S^2$, then Σ is incompressible if and only if Σ is π_1 injective.*

Remark. This fails for 1-sided surfaces. For example, $L(6, 1)$ contains a klein bottle K such that there is no simple, essential curve on K bounding an embedded disk.

We proved Dehn's Lemma in the previous class.

Proof of Corollary 1.2. Assume that Σ is π_1 injective. Let M_0 be M cut along Σ and let $M_1 = M_0 \backslash \partial M$. Consider a loop γ on Σ which is null homotopic in M. Let f be the null homotopy. We can assume f is transverse to Σ. By a slight homotopy of f, we can also make sure that $f^{-1}(\Sigma) \cap N = \partial D$ where D is bounded by γ and N is a neighborhood of ∂D (here, we use 2-sidedness). So, $f^{-1}(\Sigma)$ consists of loops in D. Consider an innermost loop of $f^{-1}(\Sigma)$. If it is inessential (trivial in $\pi_1 \Sigma$) we can homotope f to remove it. Replace f with f_1, which is the restriction of f to the innermost essential loop in $f^{-1}(\Sigma)$. Then we have, $f_1 : (D, \partial D) \to (M_1, \partial M_1)$ with $f_1|_{\partial D}$ nontrivial in π_1. So we can apply the Loop Theorem to get an embedding $f_1' : (D, \partial D) \to (M_1, \partial M_1) \to (M, \Sigma)$. This shows that π_1 injectivity of Σ implies that Σ is incompressible.
The other direction is easy. □

The outline of the proof of the Loop Theorem is as follows

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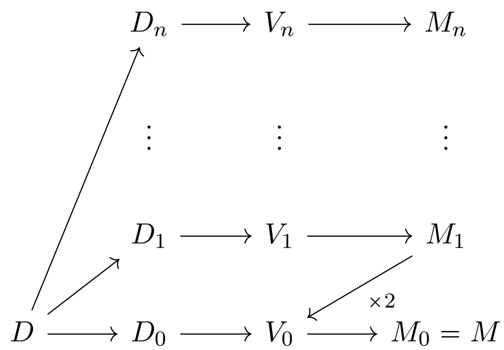
1) Build a tower of double covers



Here, D_0 is the image of f and V_0 is regular neighborhood. $M_1 \to V_0$ is a connected double cover of V_0 if such a cover exists. Lifting f, we define the other layers of the tower similarly. We must show that this tower terminates.
2) Find a nice embedded disk in V_n whose boundary is still nontrivial in π_1 when pushed down to M.
3) Push disk down the tower, resolving intersections at each stage (making sure to preserve nontriviality in π_1).

Proof. Step 1: Triangulate M. The simplicial approximation theorem implies that there is a triangulation T of D, a subdivision of M and a homotopy of pairs $f \simeq f_0$ such that f_0 is simplicial with respect to these triangulations. Let N be the number of simplices in T. We have $f_0(D) = D_0$ and that V_0 is a regular neighborhood of D_0 (a regular neighborhood is the union of simplices in the second barycentric subdivision which meet D_0). If V_0 has a connected double cover, choose one $M_1 \to V_0$. We have a lift $f_1 : (D, \partial D) \to (M_1, \partial M)$ of f_0. Let $D_1 = f_1(D)$ and let V_1 be the regular neighborhood of D_1. So long as V_i has a connected double cover, we can repeat this process.
Claim: D_i has a triangulation making $f_i : D \to D_i$ is simplicial with respect to T and the restrictions of covers $D_i \to D_{i-1}$ are also simplicial.
Each p_i indentifies some pair of simplices so $N \geqslant \#\text{simplices}(D_i) > \#\text{simplices}(D_{i-1})$. This shows that the tower terminates.
Step 2:

Lemma 1.3. *∂V_n is a union of spheres.*

Proof of Lemma. V_n has no connected double cover so $H^1(V_n; \mathbb{Z}/2\mathbb{Z}) = 0$ (there is no surjection $pi_1(V_n) \to \mathbb{Z}/2\mathbb{Z}$). For the same reason, $H_1(V_n; \mathbb{Z}/2\mathbb{Z}) = 0$. By Poincare Duality, $H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) = 0$. The long exact sequence of a pair gives

$$0 = H_2(V_n, \partial V_n; \mathbb{Z}/2\mathbb{Z}) \to H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) \to H_1(V_n; \mathbb{Z}/2\mathbb{Z}) = 0$$

so $H_1(\partial V_n; \mathbb{Z}/2\mathbb{Z}) = 0$ and ∂V_n has no nonsphere components. □

Let $p : V_n \to M$ be concatenation of double covers and inclusions. Let F_i be the component of $p_i^{-1}(\partial M)$ containing $f_i(\partial m)$ and let n be the top level of the tower. The lemma implies that F_n is planar.
Fact 1: $\pi_1(F_n)$ is normally generated by the boundary components of F_n.
Fact 2: $K_n = \ker(\pi_1(F_n) \to \pi_1(\partial M)) \neq \pi_1(F_n)$
So, there is a boundary component, α, of F_n whose image in $\pi_1(\partial M)$ is nontrivial. α bounds an embedded disk in ∂V_n. □