3-MANIFOLDS

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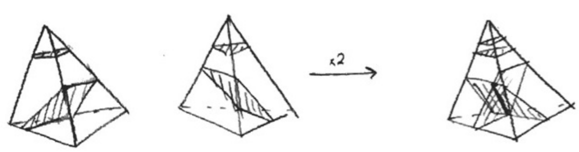
1. The Loop Theorem

Theorem. *Let M be a 3-manifold with $f : (D, \partial D) \to (M, \partial M)$ so that $f_*(\partial D) \neq 1$ in $\pi_1(\partial M)$. Then there is an embedded disk with the same property.*

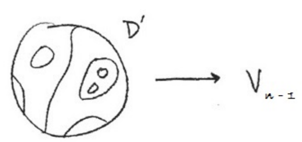
Proof. Last time we built the tower and found a nice disk at the top of the tower. That is, we found an embedding $g_n : (D, \partial D) \to (V_n, \partial V_n)$ where V_n is a space that has no double cover. We define $F_i = (p_1 \circ \cdots \circ p_i)^{-1}(\partial M)$ where $p_i : M_i \to V_{i-1}$ the double cover of V_i. If we define $N_i \subset \pi_1 F_i$ to be the kernel of $(p_1 \circ \cdots \circ p_i)_* : \pi_1(F_i) \to \pi_1(M)$, then recall that we ensured $[g_n|\partial D] \notin N_n$. Thus the image of the boundary of our embedding is nontrivial in $\pi_1(M)$ if pushed down the tower. We will now use g_n to find an embedding g_{n-1} of D into V_{n-1} with the same property. Repeating this step n times will give us an embedding g_0 of D into $M_0 = M$ which satisfies the requirements of the theorem.

Consider the map $p_n \circ g_n : D \to V_{n-1}$ which is a 2-1 cover when restricted to its image. We will isotope $D' = g_n(D)$ to be a normal surface, and then will change the map $p_n \circ g_n$ to get en embedding $g_{n-1} : D \to V_{n-1}$ as desired. To start, the image of a normal surface under a 2-fold cover has transverse self-intersections after an isotopy upstairs. In particular, isotope the surface so the triangles and quadrilaterals are affine, and the intersections with the 1-skeleton don't collide.



Curves of the self-intersection will pass through faces (and not edges) of simplices in V_{n-1}. In fact, each self-intersection looks like an X-bundle over a 1-manifold.

Now we need to show we can resolve the self-intersections of D' inside V_{n-1} while keeping $[\partial D'] \notin N_{n-1}$.



The preimage of the curves of self-intersection.

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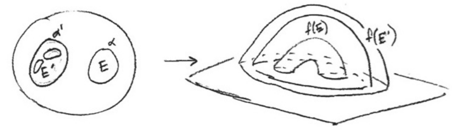
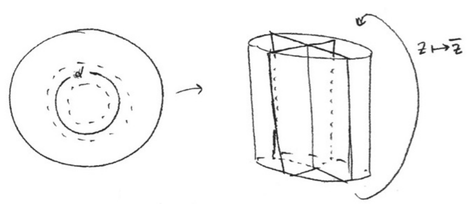
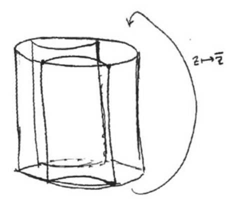
We will start by resolving any loops of self-intersection. Choose an innermost loop α of the self-intersection of D'. If there is a second loop α' with $p_n(\alpha') = p_n(\alpha)$, we can replace the disk bounded by α' with a parallel copy of the disk bounded by α. This gives a new disk with the same boundary but fewer self-intersections.



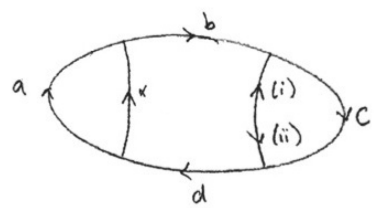
If there is no such α', then f is 2-1 on α. This means we have the following local picture:



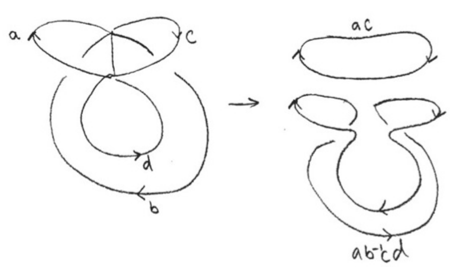
Change the map on a neighborhood of the α as shown below to remove the self-intersection along α.



Next we resolve any arcs of self-intersection. If we choose an outermost arc α of self-intersection then there will be another arc α' with $f(\alpha') = f(\alpha)$. There are two cases for the orientations:



Label the subarcs of $\partial D'$ as above. We know that $abcd \notin N_n$ since $abcd$ is the boundary of D'. In case (i), either ac or $ab^{-1}cd^{-1}$ must also not be an element of N_n.

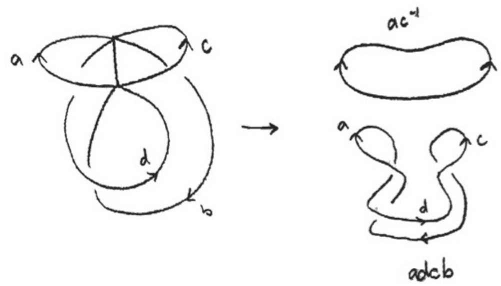


This is because we can write

$$abcd = ac[(ab^{-1}cd^{-1})^{-1}ac]^d = acd^{-1}dc^{-1}ba^{-1}acd$$

so if ac and $ab^{-1}cd^{-1}$ were both in N_n then $abcd$ would be as well. So we get a new disk with nontrivial boundary in ∂M and fewer self-intersections.

In case (ii), we have the following picture:



By the same argument as above, either ac^{-1} or $adcb$ must bound a disk with fewer self-intersections and lies outside of N_n because

$$abcd = ac^{-1}[(ac^{-1})^{-1}adcb]^{cd}.$$

This allows us to repeat until we find an embedded disk D in V_{n-1} with $[\partial D] \notin N_{n-1}$. We continue pushing D down the tower until we reach $V_0 \subset M$ and have an embedded disk in M with $[\partial D] \notin \pi_1(M)$, as desired. □

2. Applications of the Loop Theorem

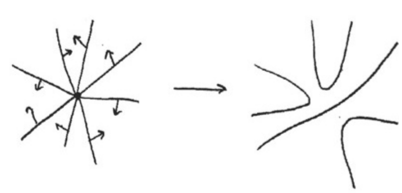
Some applications we mentioned before:

- Dehn's lemma
- Equivalence of incompressiblility and π_1-injectivity for 2-sided surfaces (not D^2, S^2).

Proposition. *Let M be a 3-manifold, $x \in H_2(M)$ or $x \in H_2(M, \partial M)$. Then x can be represented by an embedded surface. Moreover, if M is orientable then we can choose the surface so that all components are π_1-injective.*

Proof. Triangulate M and realize x as a simplicial 2-cycle (i.e. a sum of oriented 2-simplices). We want to desingularize the 2, 1, and 0-skeletons.

Start with the 2-skeleton: If a given 2-simplex occurs n times, then replace with parallel copies only intersecting in their boundary. Now any singular 1-simplex must have the same number of clockwise and counter-clockwise oriented leaves of the 2-skeleton. By matching leaves, we can resolve all the singularities.



Finally a neighborhood of a singular 0-simplex looks like a cone on some 1-manifold. Remove a neighborhood of the 0-simplex and fill it in with disjoint disks. Thus we have an embedded surface representing x.

For the moreover, note the surface we create is orientable. Thus if M is orientable then it's 2-sided and we can compress until components are π_1-injective. $\square$

Exercise: (Proof or counterexample) If the original immersed 2-cycle x is represented by a map of a closed surface S into M, then the Euler characteristic of the de-singularized embedded representation of x is bounded by the Euler characteristic of S.