THREE-MANIFOLDS NOTES (LECTURE 16)

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Applications of the Loop Theorem

In order to prove an important result of the Loop theorem, we need to use the following lemma:

Lemma (Half Lives, Half Dies)**.** *Let F be a field, and let M be a compact, F-orientable 3-manifold. Then*

$$\dim_F \Big(\ker \big(H_1(\partial M; F) \to H_1(M; F) \big) \Big)$$

$$= \dim_F \Big(im \ \big(H_2(M, \partial M; F) \to H_1(\partial M; F) \big) \Big)$$

$$= \frac{1}{2} \dim_F \big(H_1(\partial M; F) \big).$$

Proof. By Poincare Duality, we obtain isomorphisms $H^1(M; F) \cong H_2(M, \partial M; F)$ and $H_1(M; F) \cong H^2(M, \partial M; F)$. Also since $H_0(\partial M; F)$ is free, by the Universal Coefficient Theorem we have $H^1(\partial M; F) \cong H_1(\partial M; F)$. Consider the long exact sequences of the pair $(M, \partial M)$ in homology and cohomology:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{j_*} & H_2(M, \partial M; F) & \xrightarrow{\partial} & H_1(\partial M; F) & \xrightarrow{i_*} & H_1(M; F) & \xrightarrow{j_*} & \cdots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \xrightarrow{j^*} & H^1(M; F) & \xrightarrow{i^*} & H^1(\partial M; F) & \xrightarrow{\delta} & H^2(M, \partial M; F) & \xrightarrow{j^*} & \cdots .
\end{array}$$

Since i^* is dual to i_* and δ is dual to ∂, we have that

$$\dim_F (\ker i_*) = \dim_F (\text{coker } i^*) = \dim_F (\text{coker } \partial) = \dim_F (\ker \delta)$$

and

$$\dim_F (H_1(\partial M; F)) = \dim_F (\text{coker } \partial) + \dim_F (\ker i_*)$$

$$= \dim_F (H_1(\partial M; F)) - \dim_F (\text{im } \partial) + \dim_F (\ker i_*),$$

hence:

$$\dim_F(\text{im }\partial) = \dim_F(\ker i_*) = \frac{1}{2}\dim_F\big(H_1(\partial M; F)\big).$$

□

Corollary. $\mathbb{R}P^2$ *does not bound a compact 3-manifold.*

Theorem. *Let M be a compact, prime, connected, orientable 3-manifold with $\pi_1(M) \cong \mathbb{Z}$. Then M is either $D^2 \times S^1$ or $S^2 \times S^1$.*

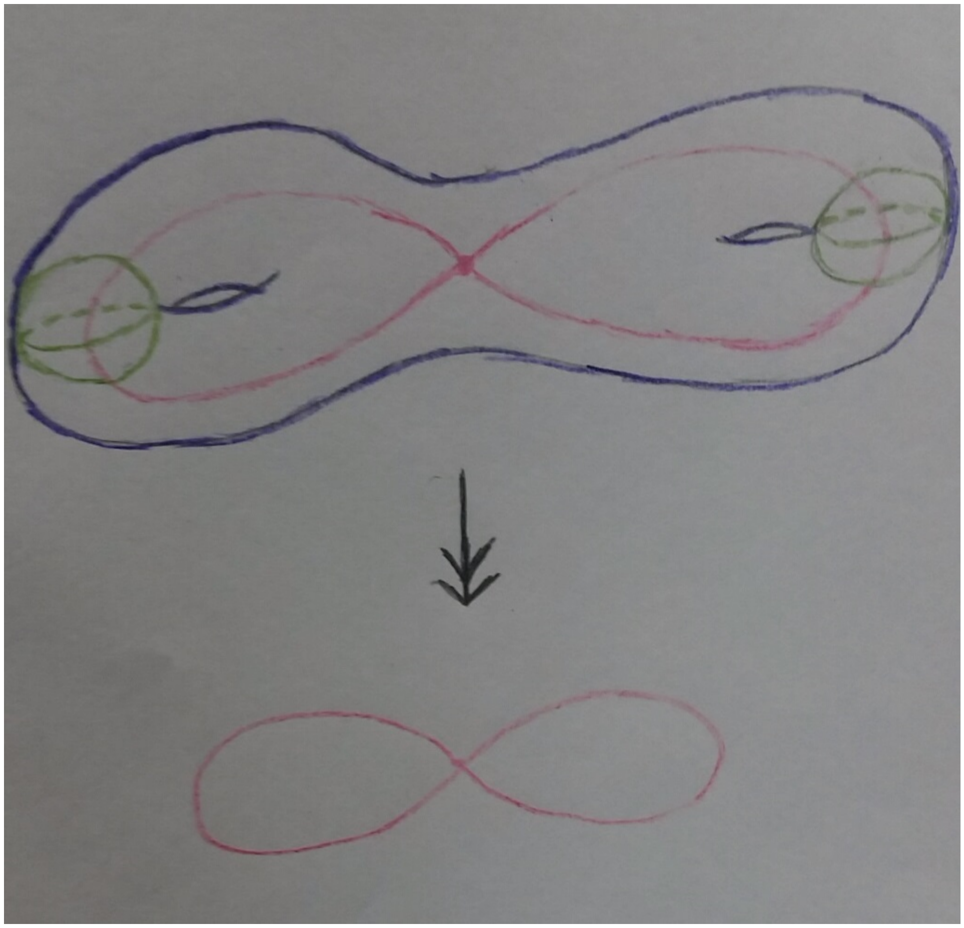
Proof. **Case 1:** Suppose $\partial M = \emptyset$. By Poincare Duality and the Universal Coefficient Theorem,

$$\mathbb{Z} = H_1(M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong H_2(M; \mathbb{Z}).$$

Let x be a generator of $H_2(M; \mathbb{Z})$, and choose a surface Σ with a π_1-injective map $\varphi : \Sigma \hookrightarrow M$ such that $[\Sigma] = x$. Since $\pi_1(M) \cong \mathbb{Z}$, Σ must be a union of 2-spheres. We claim that a choice of Σ with a minimal number of components must be connected. In order to prove this claim, we will use the following fact:

Fact. Given connected 2-sided surfaces $\Sigma_1, \ldots, \Sigma_n$ embedded in a 3-manifold M, there is a π_1-surjective map from M to the dual graph consisting of one vertex for each component of $M \setminus (\cup_i \Sigma_i)$ and one edge for each Σ_i.

Since $[\Sigma]$ generates $H_2(M; \mathbb{Z})$ and assuming Σ has a minimal number of components, Σ must be non-separating. If Σ contained more than one component, the above fact would imply the existence of a surjection $\pi_1(M) \twoheadrightarrow F_2$ onto the free group on 2 generators, a contradiction.



Hence we can assume that Σ is a 2-sphere. Now if we take two copies of Σ in M and tube them together, this gives us a separating 2-sphere $S \subset M$. Since M is prime, S must bound a 3-ball. Thus $M \approx S^2 \times S^1$.

Case 2: Suppose $\partial M \neq \emptyset$. We claim that ∂M cannot contain a 2-sphere by examining two cases: if M were prime but not irreducible, then since M is orientable we would have $M \approx S^2 \times S^1$. But this is clearly a contradiction since $S^2 \times S^1$ has empty boundary. If M is irreducible, then a 2-sphere $S \subset \partial M$ must bound a ball, implying that $M \approx B^3$. But this contradicts our assumption that $\pi_1(M) \cong \mathbb{Z}$.

Since every component C of ∂M is a compact surface and ∂M cannot contain a 2-sphere, we must have that $\dim_{\mathbb{Q}}\big(H_1(C;\mathbb{Q})\big) \geq 2$ for every component $C \subseteq \partial M$. So in particular,

$$\dim_{\mathbb{Q}}\big(H_1(\partial M;\mathbb{Q})\big) \geq 2.$$

Since $\pi_1(M) \cong \mathbb{Z}$, we have that $\dim_{\mathbb{Q}}\big(H_1(M;\mathbb{Q})\big) = 1$. So by the Half Lives Half Dies lemma,

$$\dim_{\mathbb{Q}}\big(H_1(\partial M;\mathbb{Q})\big) \leq 2,$$

hence

$$\dim_{\mathbb{Q}}\big(H_1(\partial M;\mathbb{Q})\big) = 2.$$

Therefore $\partial M \approx \mathbb{T}^2$. This implies that the inclusion $\pi_1(\partial M) \to \pi_1(M)$ has non-trivial kernel, so that ∂M is compressible. By surgering ∂M along a compressing disk $D \subset M$, we obtain a separating 2-sphere $S \subset M$ which must bound a ball $B \subset M$ because M is prime. Thus $M \approx D^2 \times S^1$.

□

Theorem. *Let M be a compact, connected, simply-connected, closed 3-manifold. Then M is either a homotopy 3-ball or a homotopy 3-sphere, possibly with punctures.*

Proof. **Case 1:** Suppose $\partial M = \emptyset$. Since M is simply-connected, by Hurewicz's Theorem, Poincare Duality, and the Universal Coefficient Theorem,

$$\pi_2(M) \cong H_2(M) \cong H^1(M) \cong H_1(M) \cong 0,$$

and

$$\pi_3(M) \cong H_3 \cong \mathbb{Z}.$$

This gives us a degree one map $\varphi : S^3 \to M$ which induces isomorphisms on homology. By Whitehead's Theorem, this gives us a homotopy equivalence $M \simeq S^3$.

Case 2: Suppose $\partial M \neq \emptyset$. Then ∂M is a union of 2-spheres. Capping off ∂M with balls gives us a homotopy 3-sphere. Thus M is a homotopy 3-sphere with n punctures, where n is the number of components of ∂M.

□

The Sphere Theorem

Theorem. *Let M be a 3-manifold with $\pi_2(M) \ncong 0$. Then either:*

1. *M contains an embedded 2-sphere Σ with $\Sigma \neq 0$ in π_2, or*
2. *M has a 2-sided $\mathbb{R}P^2$ with a double cover which has nontrivial π_2.*

Before we prove the Sphere Theorem, we'll first prove the following corollary:

Corollary. *Let M be a connected, compact, irreducible, orientable 3-manifold, and let $G = \pi_1(M)$. Then:*

1. *if G is infinite, then M is a $K(G,1)$.*
2. *if G is finite, then either M is a homotopy 3-ball, or its universal cover $\tilde{M}$ is a homotopy 3-sphere.*

Proof. By the Sphere theorem, $\pi_2(M)$ is trivial. Hence $\pi_2(\tilde{M}) \cong \pi_2(M) \cong 0$. So by Hurewicz's Theorem, $\pi_3(\tilde{M}) \cong H_3(\tilde{M})$.

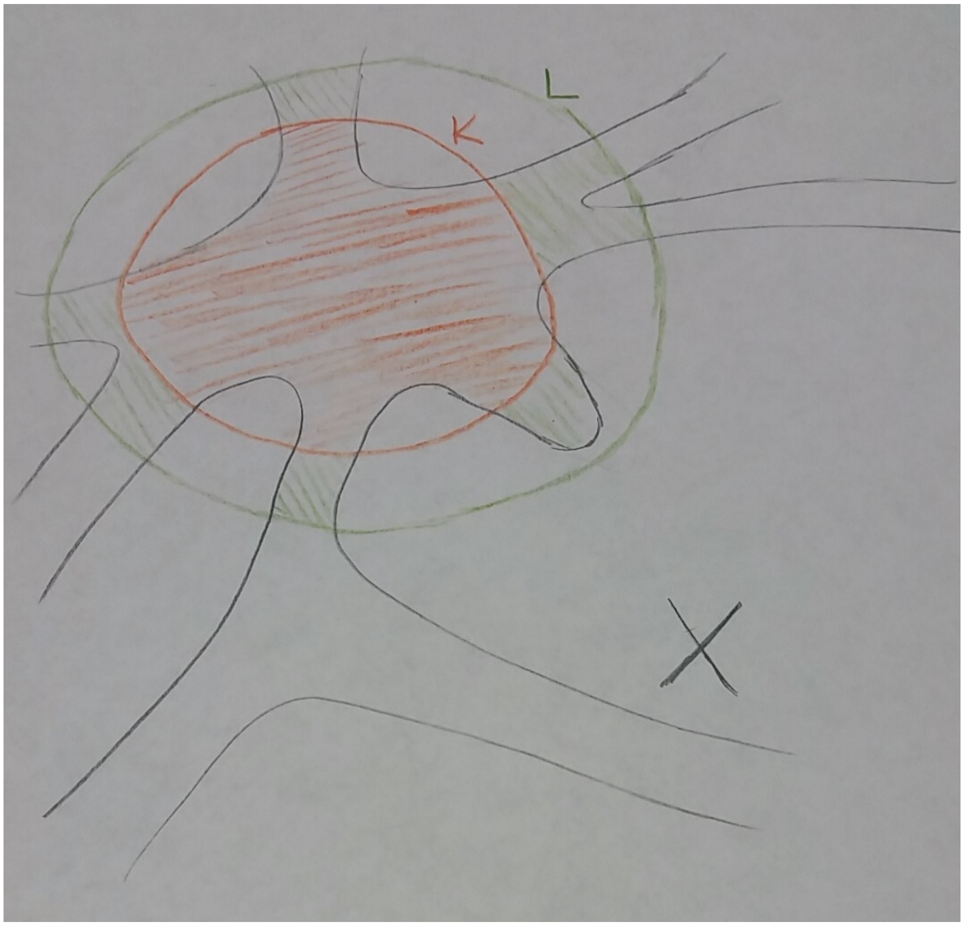
1. Suppose $G = \pi_1(M)$ is infinite. Since $\tilde{M}$ is non-compact, $H_3(\tilde{M}) \cong 0$. Similarly $\pi_i(\tilde{M}) \cong H_i(\tilde{M}) \cong 0$ for all $i \geq 4$. Hence $\tilde{M}$ is contractible, and so by the Geometrization Theorem, $\tilde{M} \cong \mathbb{R}^3$.
2. In the case where G is finite, $\partial\tilde{M}$ must be a union of 2-spheres. Since $\pi_2(\tilde{M}) \cong 0$, $\partial\tilde{M}$ must actually be a single 2-sphere. So by a previous theorem, $\tilde{M}$ is a homotopy 3-ball. And since the covering map $\partial\tilde{M} \to \partial M$ is one-sheeted, ∂M must be a single 2-sphere, implying that $\tilde{M} = M$ is a homotopy 3-ball.

□

In order to prove the Sphere Theorem, we will use the following result from geometric group theory:

Theorem (Stalling's Theorem)**.** *Let G be a finitely-generated group, and suppose that G acts properly and cocompactly on a space X with $ENDS(X) > 1$. Then G is either an amalgamated free product $A *_C B$, or an HNN extension $A*_C$ with C finite.*

Definition. Let X be a topological space, and let $\{K \subset X\}$ be a directed set of compact subsets of X, ordered by inclusion. This gives us maps $\pi_0(X \setminus L) \to \pi_0(X \setminus K)$ for $K \subseteq L$. We define $ENDS(X) := \varprojlim\{\pi_0(X \setminus K_\alpha)\}$.


L
K
X