3-MANIFOLDS

JASON MANNING
SCRIBED BY OLIVER WANG
APRIL 4TH, 2016

1. Sphere Theorem

Lemma (Lemma 1). *If Σ is an embedded S^2 in a 3-manifold M and $[\Sigma] = 0$ in $\pi_2(M)$, then Σ bounds a homotopy 3-ball.*

Proof. Let $\widetilde{M}$ be the universal cover of M. Then we get the following diagram

$$\begin{array}{ccc} & & \widetilde{M} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ \Sigma & \xrightarrow[f]{} & M \end{array}$$

where $\tilde{f}$ is a lift of f. Then, $\pi_2(\widetilde{M}) = \pi_2(M)$ and the Hurewicz theorem gives $\pi_2(\widetilde{M}) = H_2(\widetilde{M})$. So, there is some lift $\tilde{\Sigma}$ of Σ to $\widetilde{M}$ that is nullhomologous. Therefore, $\tilde{\Sigma}$ must bound a compact 3-submanifold $\widetilde{N}$. By rechoosing $\tilde{\Sigma}$, we can assume $\widetilde{N}$ contains no other lift of Σ. Van-Kampen's theorem implies $\pi_1(\widetilde{M}) = \pi_1(\widetilde{N}) * \pi_1(N') = \{1\}$ since $\widetilde{N} \cap N' = S^2$. Because $\widetilde{N}$ is simply connected, $\widetilde{N}$ is a homotopy 3-ball.
$\widetilde{N} \to N \subseteq M$ is a 1 sheeted cover so N is a homotopy 3-ball. □

Lemma (Lemma 2). *If Σ is a 2-sided $\mathbb{R}P^2 \subseteq M$ and $\tilde{\tilde{\Sigma}} \to \Sigma$ is the orientation double cover, then $[\tilde{\tilde{\Sigma}}] \neq 0$ in $\pi_2(M)$.*

Proof. Suppose otherwise. Then, we get a commuting diagram where the top row consists of orientation double covers.

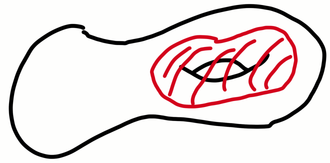
$$\begin{array}{ccc} \tilde{\tilde{\Sigma}} & \xrightarrow{\tilde{f}} & \widetilde{M} \\ \downarrow & & \downarrow p \\ \Sigma & \xhookrightarrow{f} & M \end{array}$$

p_* is a π_2 isomorphism. Suppose $[\tilde{f}] = 0$ in $\pi_2(\widetilde{M})$. Lemma 1 gives a homotopy 3-ball $\widetilde{N} \subseteq \widetilde{M}$ with $\partial \widetilde{N} = \tilde{\tilde{\Sigma}}$. Let $N = p(\widetilde{N})$. Then, $\partial N \cong \mathbb{R}P^2$, contradicting half lives-half dies. □

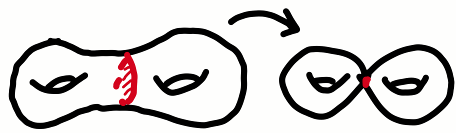
Proposition 1.1. *If the sphere theorem holds for compact 3-manifolds with incompressible (or empty) boundary, then the sphere theorem holds.*

Proof. Suppose $f : S^2 \to M$ is nontrivial in $\pi_2 M$ and simplicial. Let M_0 be a regular neighborhood of $f(S^2)$. If some ∂-component of M_0 is not π_1-injective (i.e. incompressible in M), compress the boundary until it is.

Date: April 4th, 2016.



Compressing outward is fine. Suppose ∂M_0 compresses inward.



Compressing along the compression disk shows, $M_0 \simeq A_1 \bigvee A_2$ where at least one of the A_i has nontrivial π_2 and $f = (\alpha, \beta) \in \pi_2(A_1) \bigoplus \pi_2(A_2)$. Here, either $\alpha \neq 0$ or $\beta \neq 0$. Assume $\alpha \neq 0$. Then, we can replace f by α and M_0 by A_1 (this may change the element of $\pi_2(M)$ but we only care that the element is nontrivial).

Finitely many compressions are used to give M_1 with π_1-injective ∂. Cap off 2-sphere boundaries with homotopy 3-balls to get M_2 if the sphere is trivial in π_2. Otherwise, stop: we have found an embedded essential 2-sphere/

Assuming the sphere theorem for compact manifolds with incompressible boundary, we want to find $S^2 \cong \Sigma \subseteq M_2$ so $[\Sigma] \neq 0$ in $\pi_2(M_2)$.

Claim: $[\Sigma] \neq 0$ in $\pi_2(M)$.

Proof of claim: If $[\Sigma] = 0$ in $\pi_2(M)$, there is (by Lemma 1), an embedding $(B^3, \partial B^3) \xrightarrow{\varphi} (M, \Sigma)$ which has image not entirely in M_2 where B is a homotopy 3-ball. But, $\varphi(B)$ cannot contain any ∂-component of M_2, so $\varphi(B) \subseteq M_2$. □

Compact manifolds are nice to work with because the structure of the fundamental group is reflected in large scale properties of the universal cover.

Ends: Let X be a locally compact metric space and let $K \subseteq L$ be compact subsets of X. We get a map $\pi_0(X \backslash L) \to \pi_0(X \backslash K)$.

Define

$$\text{Ends}(X) := \varprojlim \{\pi_0(X \backslash L) \to \pi_0(X \backslash K)\}$$

and

$$e(X) := \sup\{\# \text{ of unbounded somponents of } S \backslash K | K \text{ compact}\}$$

Then, $\text{Ends}(X)$ is infinite if and only if $e(X)$ is infinite. Otherwise, $e(X) = \#\text{Ends}(X)$.

Example. 1. If X is compact, then $\text{Ends}(X) = \varnothing$.
2. $\text{Ends}(\mathbb{R}) = \{\pm\infty\}$ and $e(\mathbb{R}) = 2$.
3. Ends(Regular 3-valent tree) = Cantor set.
4. $e(\mathbb{R}^2) = 1$.

We will show that, if M is compact and $\pi_2(M) \neq 0$, then $e(\widetilde{M}) \geqslant 2$.

Exercise 1.1. 1. If X is simplicial, then $e(X) = e(X^{(1)})$.

2. If Γ_1 and Γ_2 to locally finite graphs, both of which admit proper, cocompact G-actions, then $e(\Gamma_1) = e(\Gamma_2)$. So, for G finitely generated, $e(G) := e(\text{Cayley graph of } G)$ is well-defined.