THREE-MANIFOLDS NOTES

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Last Time. We reduced the proof of the Sphere Theorem to the case where M is compact and has incompressible (possibly empty) boundary.

Proof Ingredients

Theorem (Stallings' Ends Theorem)**.** *If G is a finitely-generated group with at least two ends, then G is either a nontrivial amalgam $G = A *_C B$ or an HNN extension $G = A*_C$ where C is finite.*

Alternatively, if G has at least two ends, then there is a cocompact global-fixed-point-free action of G on a tree with finite edge stabilizers.

Proposition 1. *If M is a compact 3-manifold with incompressible boundary and $\pi_2 M \neq 0$, then the universal cover $\tilde{M}$ of M (and hence $\pi_1 M$) has at least two ends.*

Proposition 2. *If M is a compact 3-manifold and $\pi_1 M$ acts cocompactly on a tree without global fixed points, then there's a (homotopically) essential 2-sided surface Σ in M with π_Σ contained in an edge stabilizer.*

We'll prove the two propositions later. For now, let's see how they are used to establish the Sphere Theorem.

Proof of Sphere Theorem. We have a compact 3-manifold M with incompressible boundary and $\pi_2 M \neq 0$. By Proposition 1, $\pi_1 M$ has at least two ends, and so it follows from Stallings' Ends Theorem that $\pi_1 M$ acts cocompactly and global-fixed-point-freely on a tree with finite edge stabilizers. From Proposition 2 we obtain a essential 2-sided surface Σ in M whose fundamental group lives in on of the finite edge stabilizers. It follows that $\pi_1 \Sigma$ is finite, so Σ is either a disk, a 2-sphere, or an $\mathbb{R}P^2$. Since ∂M is incompressible, Σ is not a disk, and so we're done by Lemmas 1 and 2 from last class. □

Graphs of Groups

Consider a graph Γ along with a group G_v for each vertex v of Γ and a group G_e for each edge e of Γ, as well as injective homomorphisms $G_e \to G_v$ whenever the edge e is incident to the vertex v. We call Γ a *graph of groups*.

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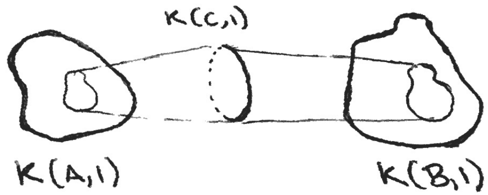
Example. Consider graph consisting of a single edge labeled by a group C and with endpoints labeled by groups A and B. This "corresponds to" the amalgam $A *_C B$ (we'll see how shortly). Similarly an HNN extension $A*_C$ corresponds to a graph consisting of on vertex labeled A and an edge labeled C with both endpoints at the vertex.

A graph of groups Γ gives rise to a topological space as follows. Construct Eilenberg-Maclane spaces $K(G_v, 1)$ and $K(G_e, 1)$ for each vertex v and edge e. For each edge e incident to a vertex v, we can construct a map $K(G_e, 1) \to K(G_v, 1)$ realizing homomorphism $G_e \to G_v$, and we glue $K(G_e, 1)$ to $K(G_v, 1)$ using the mapping cylinder of this map. The result of all of the gluings is a space (call it $X(\Gamma)$) called a *graph of spaces*. Notice that $X(\Gamma)$ has a natural surjection onto Γ.



The *fundamental group of a graph of groups* Γ can be taken to be $G = \pi_1 X(\Gamma)$. This group admits natural injections $G_v \to G$ and $G_e \to G$ for each vertex v and edge e of Γ, and G acts on a tree with quotient Γ, edge stabilizers the G_e, and vertex stabilizers the G_v.

For a more careful development and explanation of these concepts see *Trees* by Serre or *Topological Methods in Groups Theory* by Scott and Wall.

Proof of Proposition 2. It follows from the existence of the action of M on a tree that $\pi_1 M$ is an amalgam ($\pi_1 M = A *_C B$) or HNN extension ($\pi_1 M = A*_C$). The graph of spaces X corresponding to this expression of $\pi_1 M$ is a $K(\pi_1 M, 1)$ and from the isomorphism $\pi_1 M \approx \pi_1 K(\pi_1 M, 1)$ we obtain a map $f : M \to X$.

Composing f with the map from X to the underlying graph yields a map $\bar{f}$ from M to either a segment or a loop. After adjusting for transversality, the pullback of a midpoint of the edge in the target space is an embedded closed surface Σ_0 in M. (Alternatively, we can consider Σ_0 as the preimage of the $K(C, 1)$ lying in X.)

By construction, we have that $\pi_1 \Sigma$ lies in an edge stabilizer of $\pi_1 M$. It remains to modify f so that a component of Σ_0 is essential. If a 2-sphere component of Σ_0 bounds a (homotopy) 3-ball B, we can homotope f so that the image of B misses one of the vertex spaces, and then homotope f further to push B through the edge space and thereby remove the 2-sphere component from the preimage of the edge space.

Furthermore, if some component of Σ_0 admits a compressing disk D, then we can modify f by a homotopy supported in a neighborhood of D so that the effect on Σ_0 is a compression along D. (This may introduce additional inessential 2-sphere components to Σ_0, but those can be removed as before.) Hence we can homotope f until some component of Σ_0 is the desired essential surface. $\square$

HOMOLOGICAL APPROACH TO ENDS

Let X be a simplicial complex. Then $C^n(X)$ denotes the *singular* n*-cochains* of X and $C^n_c(X)$ denotes the *singular* n*-cochains with compact support*. The chain complex of *end cochains* $C^*_e(X)$ is defined by the following exact sequence.

$$0 \longrightarrow C^*_c(X) \longrightarrow C^*(X) \longrightarrow C^*_e(X) \longrightarrow 0$$

The cohomology of the end cochain complex, denoted $H^*_e(X)$, is called the *end cohomology* of X.

One can show using standard algebraic topology techniques that $H^*_e(X)$ coincides with the homology of the complex of *simplicial* end cochains.

Proposition 3. $e(X) = \dim H^0_e(X; \mathbb{Z}/2)$.

Sketch of Proof. Since we're using $\mathbb{Z}/2$-coefficients, a simplicial 0-cochain ω is just a set of vertices, and the coboundary $\delta\omega$ is the set of edges of X with one endpoint in ω and one endpoint not in ω.

Notice that $[\omega] \in H^0_e(X)$ if and only if $\delta\omega$ is finite, and $[\omega] \neq 0$ in $H^0_e(X)$ if and only if ω is an infinite set. It follows from this that $0 - \mathtt{cochains}$ in correspond to ends of X. □

To be continued!