# 3-MANIFOLDS NOTES

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We will complete the sketch of sphere theorem and Stallings ends theorem in the notes of this class. Last time we gave two propositions, restated as follows.

**Proposition 0.1.** *If $M$ is a compact 3-manifold with incompressible boundary and $\pi_2 M \neq 0$, then the universal cover $\widetilde{M}$ of $M$ (and hence $\pi_1 M$) has at least two ends.*

**Proposition 0.2.** *If $M$ is a compact 3-manifold and $\pi_1 M$ acts cocompactly on a tree without global fixed points, then there is a (homotopically) essential 2-sided surface $\Sigma$ in $M$ with $\pi_1\Sigma$ contained in an edge stabilizer.*

## 1. PROOF INGREDIENTS

We complete the proof of Proposition 1.

**Proof.** Last time we proved Proposition 3, i.e., $e(\widetilde{M}) = \dim_{\mathbb{Z}/2}(H^0_e(\widetilde{M};\mathbb{Z}/2))$. Now consider the long exact sequence

$$0 \to H^0_c(\widetilde{M}) \to H^0(\widetilde{M}) \to H^0_e(\widetilde{M}) \to H^1_c(\widetilde{M}) \to H^1(\widetilde{M}) \to \cdots$$

Since $\widetilde{M}$ is simply connected, we know $H^1(\widetilde{M}) = 0$. To show $e(\widetilde{M}) \geqslant 2$, we need to show $H^1_c(\widetilde{M})$ is non-zero. Poincare Duality gives

$$H^1_c(\widetilde{M}) \cong H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{Z}/2) \cong H_2(\widetilde{M}; \mathbb{Z}/2)$$

where the latter one is obtained by noticing that $\partial\widetilde{M} \hookrightarrow \widetilde{M}$ induces

$$H_2(\partial\widetilde{M}) = 0 \to H_2(\widetilde{M}) \to H_2(\widetilde{M}, \partial\widetilde{M}) \to H_1(\partial\widetilde{M}) = 0.$$

By Hurewicz theorem, we have $H_2(\widetilde{M}) \cong \pi_2(\widetilde{M}) \cong \pi_2(M) \neq 0$.
Exercise: Use simply connectedness of $\widetilde{M}$ plus $H_2(\widetilde{M}) \neq 0$ to deduce $H_2(\widetilde{M}; \mathbb{Z}/2) \neq 0$. □

Now comes the sketchy sketch of the Stallings' ends theorem.

**Theorem 1.1** (Stallings' Ends Theorem)**.** *If $G$ has at least two ends, then there is a cocompact global-fixed-point-free action of $G$ on a tree with finite edge stabilizers.*

**Proof.** Let $QG := \{A \subset G \mid \#\delta A < \infty\} = \{A \subset G \mid \#(A + Ag) < \infty\}$. Let $FG :=$ {finite subsets of $G$}. Then we have an alternative description of ends, i.e., $e(G) = \dim_{\mathbb{Z}/2}(QG/FG)$. We assume $e(G) \geqslant 2$ always.

**Lemma 1.2.** *If $A_0, A_1 \in QG$, then $\{g \in A_0 \mid gA_1 \subseteq A_0$ or $gA_1^* \subseteq A_0\}$ is equivalent in $QG/FG$ to $A_0$.*

A special case here is $e(G) = 2$. Suppose $A, A^*$ represent the two ends. For every $g \in G$, either $gA \sim A$ or $gA \sim A^*$. Let $G_0 < G$ be the subgroup fixing $[A]$. Some geometric argument will give you infinite order element $g \in G_0$. In fact we can get $\langle g\rangle$ finite index and normal in $G$. Some group theory will lead to $\#[G_0, G_0] < \infty$. Let $k = [G_0, G_0]$. A little more arguments gives $G/K \cong \mathbb{Z}$ or $G/K \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Hence $G = K{*_K}$ or $G = \widehat{K} *_K \widehat{K}$ for $[\widehat{K}, K] = 2$.

The second case is $e(G) > 2$ ($\Rightarrow e(G) = \infty$). Let $A \subset QG$ be nontrivial if $A, A^*$ are both infinite. Let $A$ be narrow if $\delta A$ is minimal among nontrivial $A$'s.

**Lemma 1.3.** *Given $g \in G$. There is a minimal (subject to requirement $g \in A$) narrow $A$ containing $g$.*

**Lemma 1.4.** *Let $A$ be a minimal narrow set containing 1 and let $g \in G$. Then at least one of the intersections $A \cap gA, A^* \cap (gA), A \cap gA^*, A^* \cap gA^*$ is finite, i.e., the partitions $G = A \sqcup A^*, G = gA \sqcup gA^*$ are nested. (seen as partitions of Ends(G), they are nested).*

The lemmas provide the correspondence between these partitions and the edges of a tree. Because $e(G) \geqslant 3$, the stabilizer of such a partition is finite. □

## 2. What's Next

Consider only orientable, irreducible, compact 3 manifolds with $\chi(M) = 0$. Then $\partial M$ is a (possibly empty) union of tori. The reason is that we want to study conditions which imply $M$ is fibered or virtually fibered.