THREE DIMENSIONAL MANIFOLDS, SPRING 2016

1. Lecture 2

1.1. **A review of some smooth topology.** As was mentioned last time, we'll be working with smooth manifolds and maps. A couple of important facts we'll use:

- (Tubular neighborhoods) If A is a smoothly and properly embedded submanifold of M, then A has a neighborhood diffeomorphic to its normal bundle.
- (Transversality) If A is a codimension α submanifold of M, and B is a codimension β submanifold, then A can be isotoped (by an arbitrarily small isotopy) so that $A \cap B$ is a smooth codimension $(\alpha + \beta)$ submanifold of M.

We'll also use some **Morse theory** (see Milnor's book [1]). Let M^n be a closed manifold. We'll say a function $f\colon M \to \mathbb{R}$ is *Morse* if

(1) f has finitely many critical values.
(2) For each critical value c, there is exactly one critical point p so that $f(p) = c$.
(3) For a critical point p, there is an open set U containing p and a chart $\phi\colon U \to \mathbb{R}^n$, so that $\phi(p) = \mathbf{0}$, and
$$f \circ \phi^{-1}(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$
The number k is independent of such a chart and is called the *index* of the critical point.

Figure 1 shows an example of a torus in 3-space, for which the height (projection to

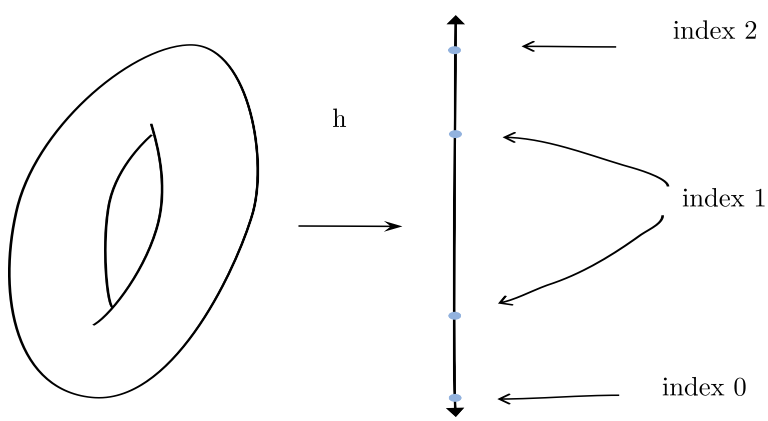

Figure 1. A torus with a Morse function on it.

the z–axis) is Morse, and points out the indices of the critical points corresponding to various critical values.

The following is proved in the first few pages of Milnor's book:

Theorem. *Any smooth function on a manifold can be approximated arbitrarily closely by Morse functions.*

One way Morse functions are used is to generate *handle decompositions* of manifolds. The idea is to consider sublevel sets of the Morse function. Fix a Morse $f\colon M \to \mathbb{R}$, where M is a manifold without boundary. Another basic result of Morse theory is the following (terminology to be defined afterward):

Theorem. *Let $[t_1, t_2] \subset \mathbb{R}$ be an interval, chosen so that neither t_1 nor t_2 is a critical value of the Morse function f Let $M_i = f^{-1}((-\infty, t_i])$, for $i = 1, 2$. Then*

(1) *If $[t_1, t_2]$ contains no critical value, then $M_2 \cong M_2$.*
(2) *If $[t_1, t_2]$ contains exactly one critical value, corresponding to a critical point of index k, then M_2 is obtained from M_1 by* attaching a k–handle to ∂M_1, *and then* smoothing the resulting corner.

An n–dimensional *k–handle* is best thought of as a "thickened k–cell". Precisely, we have

$$h^k \cong D^k \times D^{n-k}.$$

The *core* of h^k is $D^k \times \{0\}$, and the *co-core* is $\{0\} \times D^{n-k}$. The *attaching boundary* of h^k is the thickened version of the boundary of D^k: it is equal to $\partial D^k \times D^{n-k}$. To *attach a k–handle to M*, is to form the space

$$M' = M \cup_\phi h^k$$

where ϕ is a diffeomorphic embedding of the attaching boundary of h^k into ∂M. The resulting M' is not quite a smooth manifold, without some thought. It is a *manifold with corners*:

Definition 1.1. Let $Y = \{(x, y) \in \mathbb{R}^2 \mid (y \geq 0, x \geq 0) \text{ or } (y \leq 0, x \geq 1)\}$, and let $X = Y \times \mathbb{R}^{n-2}$. Let $\mathcal{G}$ be the pseudogroup of diffeomorphisms of open sets in X which extend to diffeomorphisms of $\mathbb{R}^n$. If a manifold with boundary has a $(\mathcal{G}, X)$–structure, we call it a *manifold with corners*

Exercise 1. "smooth the corners": Given a manifold-with-corners, there is a canonical way to build a smooth manifold with boundary. (hint: see the appendix to these notes of Milnor `http://faculty.tcu.edu/gfriedman/notes/milnor1.pdf`)

1.2. **Alexander's theorem.** To get us started with the prime decomposition, we must first understand whether S^3 can be a nontrivial connect sum. That it is not will follow from

Alexander's Theorem. *Let Σ be a smooth 2–sphere in $\mathbb{R}^3$. Then Σ bounds a ball.*

In class, I gave a proof of Alexander's Theorem, following Hatcher's notes fairly closely. In outline I said: Isotope Σ if necessary so that projection to the z–axis is Morse. Now do surgery using horizontal discs at heights between the critical values to cut Σ into a bunch of smaller 2–spheres, each of which clearly bounds a ball. Reverse the surgery to glue these balls together into a big one, using the following lemma:

Lemma 1.2. *Let M^3 be a manifold with boundary, and let $D \subset M$ be a properly embedded smooth disk, cutting off a ball B. Let $N = M \setminus \text{interior}(B)$. Then $M \cong N$.*

Starting with the ball corresponding to the lowest critical value, glue to a neighboring ball, and continue. Note that you may occasionally need to *remove* a ball rather than glue a new one on. I missed this point in class, but applying the procedure to the example pictured in Figure 2 illustrates the point.

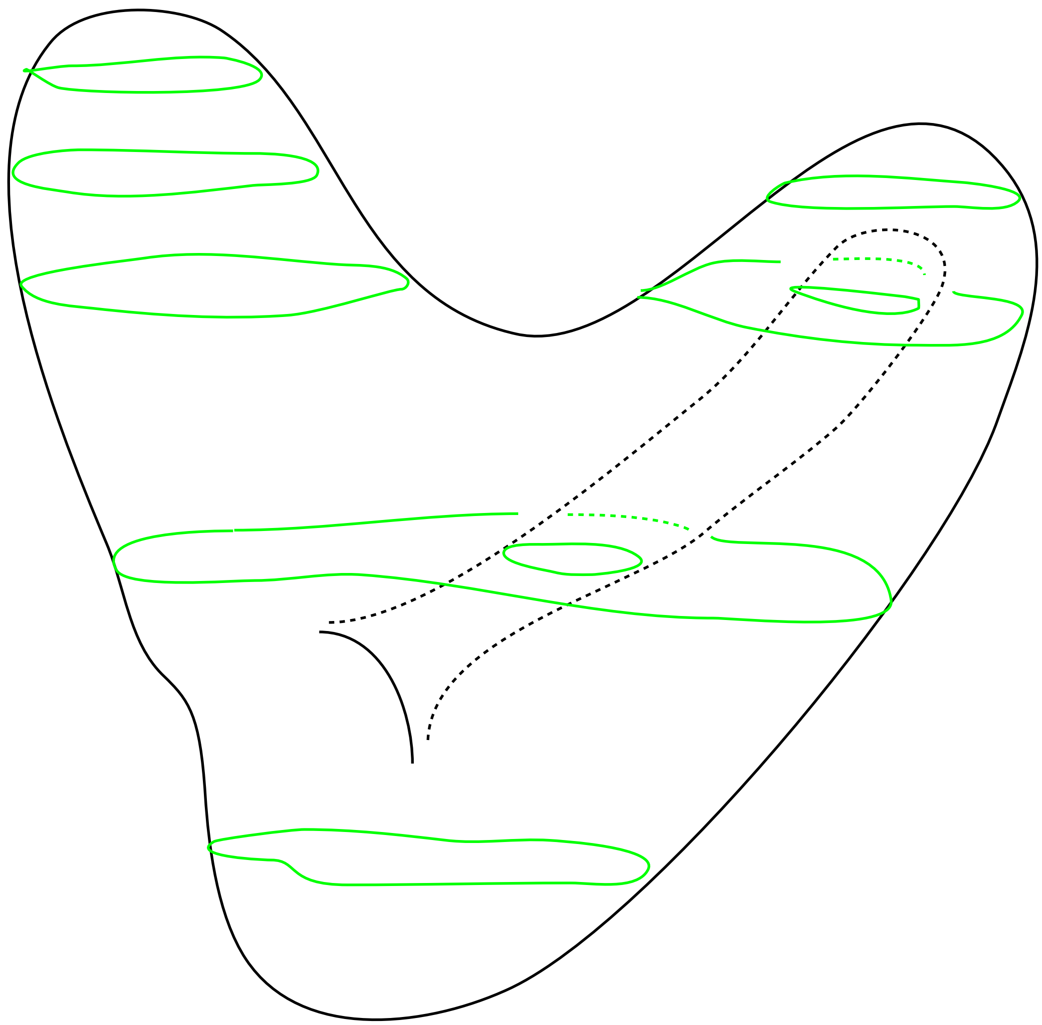

FIGURE 2. A sphere where the bounded ball is built by iteratively adding *and removing* 3–balls to/from a 3–ball. Green curves are drawn at levels between critical points of the height function.

References

[1] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.