THREE-MANIFOLDS NOTES

JASON MANNING, NOTES BY AARON PALMER, EDITED BY ZHEXIU TU

Fibered Three Manifolds

Let $\alpha \in H^1(M;\mathbb{Z})$. Then by realization theorem, $H^1(M;\mathbb{Z})$ is equivalent to the homotopy classes of maps $\varphi : M \to S^1$ (because $S^1 \cong K(\mathbb{Z}, 1)$), so let φ represent α.

Definition 1. $\alpha \in H^1(M;\mathbb{Z})$ is a *fibered* class if we can choose a representation $\varphi : M \to S^1$ to be a fiber bundle map (S^1 is the base space, $\varphi^{-1}(\theta)$ is a fiber).

In any case, for a regular (generic) point $\theta \in S^1$, $\varphi^{-1}(\theta) = (S, \partial S) \subset (M, \partial M)$ and $[S, \partial S] \in H_2(M, \partial M;\mathbb{Z})$ is Poincaré dual to $\alpha \in H^1(M;\mathbb{Z})$.

Note that $H^1(M;\mathbb{Z})$ is torsion free so $H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{Q})$.

Definition 2. For $\alpha \in H^1(M;\mathbb{Q})$, α is *fibered* if $n\alpha \in H^1(M;\mathbb{Z})$ is fibered for some $n > 0$.

For $\alpha \in H^1(M;\mathbb{Z})$ represented by $\varphi : M \to S^1$, 2α may be represented by $p \circ \varphi$ where $p : S^1 \to S^1$ is a degree 2 map.

Definition 3. For a dual class $\alpha^* \in H_2(M, \partial M;\mathbb{Q})$ we say α^* is fibered if the Poincaré dual is fibered.

Note $\beta \in H_2(M, \partial M;\mathbb{Q})$ is fibered iff $\exists n > 0$ so $n\beta = [S, \partial S]$ where S is a fiber of a fiber bundle structure on M.

Theorem 4 (Agol's Theorem). *If M is compact, orientable, irreducible, with $\chi(M) = 0$, and $\pi_1(M)$ is RFRS, and $\varphi \in H^1(M;\mathbb{Q})\backslash\{0\}$, then there is a finite sheeted cover $p : \tilde{M} \to M$ such that $p^*(\varphi)$ is a limit (with respect to norm) of fibered classes.*

Later we define *RFRS* as " residually finite rational solvable" and *Virtually Special* implies *Virtually RFRS*.

For three manifolds, $\chi(M) = \frac{1}{2}\chi(\partial M)$ and since M is irreducible and $\chi(M) = 0$, M has no sphere boundary components. Thus M has only torus boundary components.

Date: 13 Apr. 2016.

THURSTON NORM

Reference: Thurston - "A norm on the homology of 3-manifolds."

Let S be a compact connected surface. Define

$$X_-(S) = \max\{0, -\chi(S)\}. \tag{1}$$

If S is disconnected with components S_i, let

$$X_- = \sum_i X_-(S_i). \tag{2}$$

Let M be a connected, compact, orientable, irreducible 3-manifold. Let $N \subset \partial M$ be a subsurface with possibly non-empty boundary. For $\alpha \in H_2(M, N; \mathbb{Z})$ define

$$X(\alpha) = \min_{[S]=\alpha}\{X_-(S)\}. \tag{3}$$

We call S norm-minimizing if it realizes the minimum in (3).

Lemma 5. *Let $\alpha \in H_2(M, N; \mathbb{Z})$, then there is S norm-minimizing with $[S] = \alpha$ such that*

(1) *S has no compressing disk.*

(2) *If $N = \partial M$ (*), S has no boundary compression disks.*

(3) *S has no 2-sphere components.*

Proof. Let S be a norm-minimizing surface for $\alpha \in H_2(M, N; \mathbb{Z})$.

Remove homologically trivial components, so that every component is non-separating. Thus there are no sphere components of S because M is irreducible.

Suppose there is a compressing disk with boundary $\gamma \subset S'$ for S' a connected component of S. If γ is non-separating in S' then compress. This decreases $X_-(S')$ unless $\chi(S') \geq 0$. If $S' \cong T^2$ then it becomes a separating sphere and we can throw it out. The other significant case is $S' \cong A$ an annulus, in which case S' splits into two disks, which does not change homology class or Thurston norm.

The proof of 2. is similar, but deferred to next lecture to check what can be said without (*). $\square$

Proposition 6. *In above situation, $X : H_2(M, N; \mathbb{Z}) \to \mathbb{Z}$ satisfies:*

(1) $X(\alpha) \geq 0$.

(2) $X(n\alpha) = |n|X(\alpha)$.

(3) $X(\alpha+\beta) \leq X(\alpha) + X(\beta)$,

which makes X *a semi-norm on* $\mathbb{Z}^n$.

X *extends to a polyhedral semi-norm on* $\mathbb{R}^n$ *for* n *the dimension of* $H_2(M, N; \mathbb{Z})$*, i.e. the unit norm ball is a finite intersection of half-spaces.*

Proof. 1. is satisfied by definition. 2. a) $X(n\alpha) \leq |n|X(\alpha)$ follows from considering $n\alpha$ as n copies of a representative for α. b) Need to show $X(n\alpha) \geq |n|X(\alpha)$. Let $n\alpha = [S, \partial S]$ for $(S, \partial S)$ norm-minimizing. Choose $p_0 \notin (S, \partial S)$ with $p_0 \in \partial M \backslash N$ if $\partial M \neq N$. Then consider $\varphi : M\backslash S \to \mathbb{Z}/n\mathbb{Z}$ for n the number of components of $M\backslash S$ and φ the intersection number of a curve connecting p_0 to x. FIRE. To be continued. $\square$