3-MANIFOLDS

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1. Thurston Norm

Recall we defined the complexity of a surface S in a 3-manifold as

$$\chi_-(S) = \Sigma\{\max\{0, -\chi(S')\} : S' \text{ is a component of } S\}.$$

Let $N \subset \partial M^3$ be a subsurface. Then we defined the **Thurston Norm** $x : H_2(M, N; \mathbb{Z}) \to \mathbb{Z}$ by

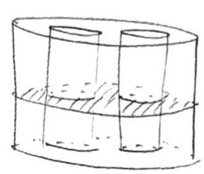
$$x(\alpha) = \min\{\chi_-(S) : [S] = \alpha\}.$$

Now we define

Definition. $(S, \partial S) \subset (M, N)$ is **essential** if

(i) S has no compressing disks
(ii) S has no boundary compressing disks into N
(iii) S is not a 2-sphere bounding a homotopy 3-ball
(iv) S is not a disk bounding a homotopy 3-ball together with a disk in N (i.e. S is not N-parallel)

Example. F a surface, $M = F \times I$, $N = \partial F \times I$. Then $F \times \{1/2\}$ is essential in (M, N), but inessential in $(M, \partial M)$.



Lemma. $\alpha \in H_2(M, N)$ *is represented by a norm-minimizing essential surface.*

Proof. Proof left as an exercise. □

Now we return to proving the properties of the function $x : H_2(M, N) \to \mathbb{Z}$ which make it a semi-norm.

(i) $x(\alpha) \in \mathbb{Z}_{\geqslant 0}$ for all $\alpha \in H_2(M, N)$
(ii) $x(n\alpha) = |n|x(\alpha)$ for all $\alpha \in H_2(M, N), n \in \mathbb{Z}$
(iii) $x(\alpha + \beta) \leqslant x(\alpha) + x(\beta)$ for all $\alpha, \beta \in H_2(M, N)$.

Proof.

(i) Proved last time.

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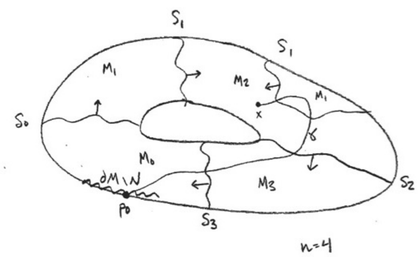
(ii) $x(n\alpha) \leqslant |n|x(\alpha)$ is clear because n times any representative of α will be a representative for $n\alpha$. Now we want to show $x(n\alpha) \geqslant |n|x(\alpha)$. We begin by choosing $(S, \partial S)$ representing $n\alpha$. Pick a basepoint in $p_0 \in M\backslash S$. If $\partial M\backslash N \neq \varnothing$, then choose $p_0 \in \partial M\backslash N$. We define $f : M\backslash S \to \mathbb{Z}/n\mathbb{Z}$ by

$$f(x) = \text{intersection number of } \gamma \cap S \text{ for any arc } \gamma \text{ from } p_0 \text{ to } x$$



If φ is Poincaré dual to S, then $f(x) = \varphi(\gamma) \mod n$. Define, for $i \in \mathbb{Z}/n\mathbb{Z}$, $M_i = \overline{f^{-1}(i)}$.

Claim: If $(\partial M\backslash N) \cap M_i \neq \varnothing$, then $i = 0$. Proof: Let γ be an arc joining p_0 to $(\partial M\backslash N) \cap M_i$. Then γ is a cycle in $H_1(M, \partial M\backslash N)$, so $\varphi(\gamma) = \langle n\alpha, \gamma\rangle = n\langle \alpha, \gamma\rangle \equiv 0 \mod n$. Therefore $i = 0$.

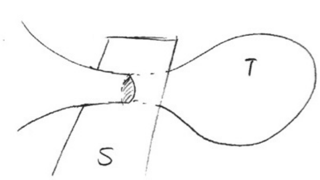
For each i, we define S_i as $M_{i+1} \cap M_i$. Then for each i we have $S_i \sqcup -S_{i-1} \subseteq \partial M_i$. For $i \neq 0$, $\partial M_i\backslash N = S_i \sqcup -S_{i-1}$. Since the S_j and S_{j+1} bound M_j for all j we have

$$[S_0] = [S_1] = \cdots = [S_{n-1}]$$

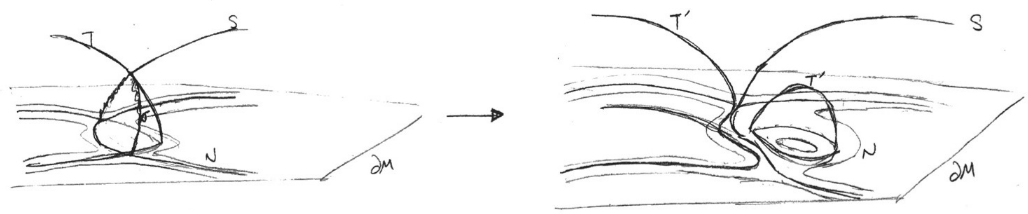
and so each $[S_i] = \alpha$ since $\Sigma[S_i] = [n\alpha]$. Therefore $x(n\alpha) \geqslant |n|x(\alpha)$, as desired.

(iii) Let $\alpha = [S, \partial S]$ and $\beta = [T, \partial T]$ for essential, norm-minimizing S and T. We can assume by a small isotopy that S and T are transverse to each other. First we will do a surgery to get $S \cap T$ essential in both S and T:

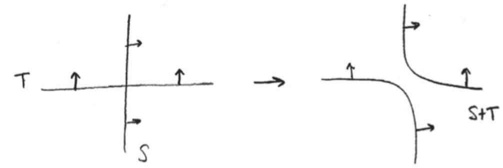
Suppose $\gamma \subset S \cap T$ is an innermost loop bounding a disk in S. Since T is essential, γ must also bound a disk in T. Surger and remove this 2-sphere component from T (we assume that M is irreducbile so this doesn't change the complexity of T). This will reduce the number of components of $S \cap T$.



Now suppose σ is an arc of $S \cap T$ which is inessential in S. Since T is essential in (M, N), σ is inessential in T. γ along with an arc a_S in N bounds a disk in S, and with an arc a_T in N bounds a disk in T. We surger T to remove γ, creating a new component of T which is a disk with boundary in N.



Now every component of $S \cap T$ is essential in both. Let $S + T$ be the surface obtained by doing a double-curve sum:



Then $\chi(S + T) = \chi(S) + \chi(T)$, so the complexity can only change if new disks or spheres were formed. They aren't because the intersections were along essential arcs and loops. Thus $x(\alpha + \beta) \leqslant x(\alpha) + x(\beta)$, as desired.

□