THREE-MANIFOLDS NOTES

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Last Time. We showed that $x : H_2(M, N) \to \mathbb{Z}$ is a seminorm. That is,

- $x(\alpha) \geq 0$ for all $\alpha \in H_2(M, N)$
- $x(n\alpha) = |n|x(\alpha)$ for all $n \in \mathbb{Z}$ and $\alpha \in H_2(M, N)$
- $x(\alpha + \beta) \leq x(\alpha) + x(\beta)$ for all $\alpha, \beta \in H_2(M, N)$.

Hence, we can extend x to a seminorm $H_2(M, N; \mathbb{Q}) \to \mathbb{Q}$ by defining $x(\alpha) = \frac{x(n\alpha)}{n}$ where $n\alpha \in H_2(M, N)$. This can be extended continuously to a seminorm

$$x : H_2(M, N; \mathbb{R}) \to \mathbb{R}.$$

A (semi)norm is determined by $B_x = \{v : x(v) \leq 1\}$, which is convex and symmetric about 0. As we will see, integrality puts strong constraints on B_x.

Remark. If M^3 is orientable, irreducible, atoroidal, and has no essential annuli or disks, then $x : H_2(M, \partial M) \to \mathbb{Z}$ satisfies $x(v) = 0 \iff v = 0$. we want to be sure that this is true after we extend to $\mathbb{R}$ as well.

Fix $x : \mathbb{Z}^n \to \mathbb{Z}$ satisfying the conditions for a seminorm, and extend it to $\mathbb{R}^n \to \mathbb{R}$.

Lemma 1. *The set* $Z = \{v \in \mathbb{R} : x(v) = 0\}$ *is spanned by* $Z \bigcap \mathbb{Z}^n$*.*

Corollary 2. *In the situation of the above remark,* $x : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ *is a norm.*

Proof of Lemma 1. Suppose $v \in \mathbb{R}^n$ satisfies $x(v) = 0$ and $v \notin \mathbb{Q}^n$. Then the ray $\{sv : s \geq 0\}$ comes arbitrarily close to integral points v_i with $x(v_i) = 0$ since x takes integral values on $\mathbb{Z}^n$. So v is approximated by rational vectors $\frac{1}{s_i}v_i$ and it is therefore in the span of those vectors. □

Theorem 3. *The set* B_x *is a finite sided polyhedra. That is, it is a finite intersection of half-spaces.*

To prove the theorem, we will need the following lemma

Lemma 4. *For any* $\alpha \in \mathbb{Z}^n - \{0\}$*, there exists a linear map* $\ell_\alpha : \mathbb{R}^n \to \mathbb{R}$ *such that*

- $\ell_\alpha(\frac{\alpha}{x(\alpha)}) = 1$
- $\ell_\alpha(\beta) \in \mathbb{Z}$ *for all* $\beta \in \mathbb{Z}^n$
- $||\ell_\alpha|| = \sup\{\ell_\alpha(v) : x(v) \leq 1\} = 1.$

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Proof of Lemma 4. For every $k \in \mathbb{Z}_{>0}$, let $B_k = \{v : x(v) \leq k\}$. Let C_k be the convex hull of integral points of B_k. Note that $B_x = B_1$ is approximated by $\frac{1}{k}C_k$ since rational points are dense. In fact, $B_x = \bigcup \left\{\frac{1}{k}C_k : k \in x(\alpha)\mathbb{N}\right\}$. The vector $k\frac{\alpha}{x(\alpha)}$ is in ∂C_k, so choose a top dimensional face of ∂C_k containing $k\frac{\alpha}{x(\alpha)}$, as in figure 1. Let $\rho_k : \mathbb{R}^n \to \mathbb{R}$ be the (integral linear) map such that this face lies in $\{v : \rho_k(v) = k\}$. Similarly, $\frac{1}{k}C_k \subseteq \{v : \rho_k(v) \leq 1\}$. So $||\rho_k|| \to 1$. Some subsequence must converge to ρ_∞ (and in fact eventually be constant) with $||\rho_\infty|| = 1$. The map ρ_∞ is our desired linear map. ☐

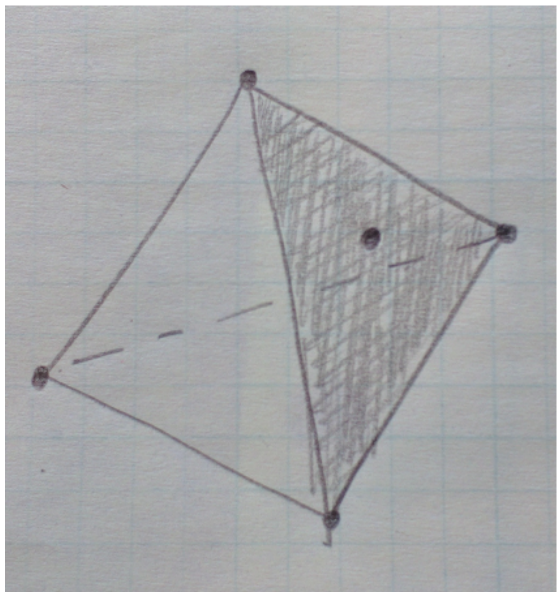

FIGURE 1. Top dimensional face

Proof of Theorem 3. Note that Z is a summand of $\mathbb{Z}^n$, so that $\overline{x} : \mathbb{Z}^n/Z \to \mathbb{Z}$ is an integral norm. Polyhedrality of $\overline{x}$ implies polyhedrality of x, so we can assume without loss of generality that $Z = \{0\}$. Now we show that lemma 4 implies the theorem. The points $\frac{\alpha}{x(\alpha)}$ with $\alpha \in \mathbb{Z}^n - \{0\}$ are dense in the boundary of B_x. Also,

$$B_x = \bigcap_{\alpha \in \mathbb{Z}^n - \{0\}} \{v : \ell_\alpha(v) \leq 1\}.$$

However, since the ℓ_α are integral and have $||\ell_\alpha|| = 1$, there are only finitely many such. ☐

Remark. For the dual norm x^*, we have B_{x^*} is also polyhedral

NEXT GOAL

Now, our next goal is to show that

$$\{\alpha \in H_2(M, N; \mathbb{Q}) : \alpha \text{ is fibred}\}$$

is the set of rational points in the union of cones on (some of the) open top dimensional faces of B_x. See figure 2.

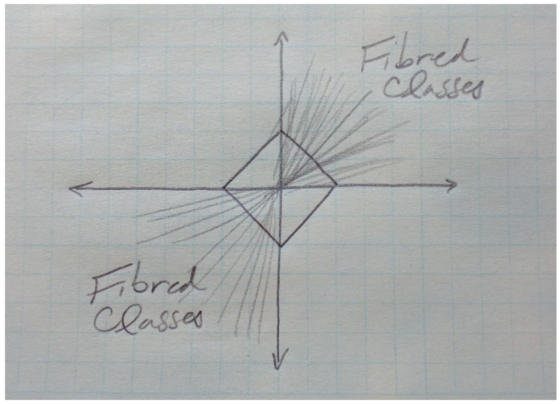

FIGURE 2. Fibred classes form cones

We will first show the following easier claim

Lemma 5. *If* $\alpha \in H_2(M, \partial M)$ *is fibred, then so is every rational class in a neighborhood of* $\{s\alpha : s > 0\}$.

Lemma 5 follows from a theorem of Tischler:

Theorem 6. *A class* $\alpha \in H^1(M; \mathbb{Z})$ *is fibred if and only if* α *is represented (in de Rham cohomology) by a nondegenerate 1-form with integral periods.*

Sketch of a Proof of Theorem 6.

($\Rightarrow$) Let $\varphi : M \to S^1$ be a map representing α. Set $\omega = \frac{1}{2\pi}\varphi^*(d\theta)$, where $d\theta$ is the 1-form on S^1. Since φ is a fibre bundle, ω is nondegenerate.

($\Leftarrow$) Reverse the process. Choose a basepoint $x_0 \in M$. Define $\varphi : M \to \mathbb{R}/\mathbb{Z}$ by

$$\varphi(x) = \int_\gamma \omega$$

where γ is any smooth path from x_0 to x. This is a fibre bundle with no critical points. (c.f. Morse Theory.) □