THREE-MANIFOLDS NOTES

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Last Time. We showed the that Thurston norm is *polyhedral*, that is, its unit norm ball is a finite sided (convex) polyhedron in $H_2(M, \partial M; \mathbb{R})$.

More Thurston Norm Stuff

Lemma 1. *Let* $(F, \partial F)$ *be a fiber of the 3-manifold* $(M, \partial M)$*. Then* F *is norm-minimizing in its homology class* $[F, \partial F] \in H_2(M, \partial M; \mathbb{R})$.

Proof. Let $(S, \partial S)$ be a *norm-minimizing* and *essential* surface in $(M, \partial M)$ which is homologous to F. If $\tilde{M}$ is the infinite-cyclic cover of M that is the pullback of the cover $\mathbb{R} \to S^1$ under the bundle map $M \to S^1$, then S lifts to $\tilde{M}$:

$$\begin{tikzcd} & \tilde{M} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ (S, \partial S) \arrow[r] \arrow[ur, dashed] & M \arrow[r] & S^1 \end{tikzcd}$$

We have $\tilde{M} \cong F \times \mathbb{R}$, so we can consider the map $(S, \partial S) \to (F, \partial F)$ resulting from projection of the lift onto the first factor. This map has degree one and must therefore be π_1-surjective, and since S is essential it must be π_1-injective. It follows that $(S, \partial S) \to (F, \partial F)$ is a homotopy equivalence of pairs, and so it follows that $\chi_-(F) = \chi_-(S)$ is minimal. $\square$

Foliations

Definition. A k-dimensional *foliation* of an n-manifold M is a decomposition of M into (not-necessarily-closed) immersed k-submanifolds (called *leaves*) such that M locally looks like $U \times \mathbb{R}^{n-k}$, where U is a coordinate chart in $\mathbb{R}^k$ and each $U \times \{pt\}$ is contained in a single leaf.

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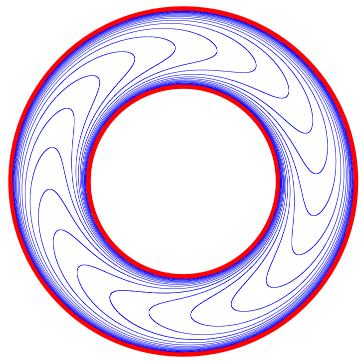
Example. A 1-dimensional foliation of an annulus[1]:



We obtain a foliation of a torus by doubling this picture along its boundary. Notice that the resulting foliation has two compact leaves and uncountably-many noncompact leaves.

Definition. A codimension-1 foliation $\mathcal{F}$ of M is called *taut* if for every leaf $\ell \subset \mathcal{F}$ there is a loop in M that meets ℓ and is transverse to the leaves of $\mathcal{F}$.

Example. The foliation of the torus described above is *not* taut: any loop that passes through one of the compact leaves must be tangent to one of the noncompact leaves.

Note that a fibration (of a 3-manifold over S^1) gives rise to a taut foliation, so the associated leaves (the fibers) are norm-minimizing by Lemma 1. More generally, it follows from a result of Gabai and Thurston that a compact leaf of a taut foliation is always norm-minimizing.

Back to Thurston Norm Stuff

Our goal is to understand the following theorem of Thurston.

Theorem. *If* $(F, \partial F) \subset (M, \partial M)$ *is a fiber with* $\alpha = [F, \partial F] \in H_2(M, \partial M)$ *and* $x(\alpha) > 0$*, then there is a top-dimensional face of* B_x *so that* α *is contained in the open cone* C *on that face. Moreover, every rational class in* C *is fibered.*

The first claim of the theorem can be proved using the following proposition.

Proposition 2. *For* $\alpha \in H_2(M, \partial M; \mathbb{Z})$ *as above, there is a class* $e \in H^2(M, \partial M; \mathbb{Z})$ *so that* $x(\beta) = \langle -e, \beta \rangle$ *for all* β *in a neighborhood of the ray spanned by* α*.*

Let $\mathcal{F}$ be the foliation coming from the fiber bundle map $\varphi : M \to S^1$. Then we obtain a tangent plane field $T\mathcal{F} \subset TM$ is given by $\ker(\omega)$ where $\omega = \varphi^*((1/2\pi)d\theta)$.

Definition. Let N be a closed oriented n-manifold, and suppose $E \to N$ a k-dimensional vector bundle over N. Let $s : N \to E$ be a generic section that is transverse to the zero section of E. Then the zero set of s is an $(n-k)$-dimensional submanifold of N which

[1]from `https://en.wikipedia.org/wiki/Foliation`

is dual to a k-dimensional cohomology class $e(E) \in H^k(N;\mathbb{Z})$. This class is called the *Euler class* of E.

When N has boundary we need to be more careful with what sections we allow. For a section $s_\partial : \partial N \to E$, we extend to a section $s : N \to E$ and get a relative Euler class $e(E, s_\partial) \in H^k(N, \partial N;\mathbb{Z})$ as above.

Proposition 3. *If Σ is a closed orientable surface, then* $e(T\Sigma) = \chi(E) \cdot [\Sigma]^*$.

Note that Proposition 3 is a special case of the Poincaré-Hopf Index Formula, since a section of $T\Sigma$ is the same as a vector field on Σ.

> **Exercise 1.** Show that the proposition hold in the relative case if we choose s_∂ to be an inward-pointing nonzero vector field.

In the context of the theorem above, from a fibration we get a foliation $\mathcal{F}$, and from $\mathcal{F}$ we get a relative Euler class $e = e(T\mathcal{F}, s_\partial) \in H^2(M, \partial M;\mathbb{Z})$, where s_∂ is a vector field on ∂M that points inwards along the leaves. Notice that, for a fibered class $\alpha \in H_2(M, \partial M)$ with (norm-minimizing) fiber $(F, \partial F) \subset (M, \partial M)$, we have

$$-\chi(F) = -\langle e, \alpha\rangle = x(\alpha)$$

by Proposition 3. This forms the basis for the following proof of Proposition 2.

Proof of Proposition 2. As suggested above, take e to be the Euler class $e(T\mathcal{F}, s_\partial)$ of the foliation induced by the fibration corresponding to α. Then we have $x(\alpha) = \langle -e, \alpha\rangle$ as above, and by linearity we have $x(\beta) = \langle -e, \beta\rangle$ for β in the ray spanned by alpha. It remains to show that the same holds in a neighborhood of the ray; by linearity it suffices to show that the equation holds in a neighborhood of α.

Let ω be a nondegenerate 1-form with $\ker \omega = T\mathcal{F}$, or (equivalently) a representative of the dual to α in de Rham cohomology. Let $\{\omega_1, \dots, \omega_l\}$ be a $\mathbb{Z}$-basis for $H^1(M;\mathbb{R})$. Then for small ε_i, the 1-form

$$\omega' = \omega + \varepsilon_1\omega_1 + \cdots + \varepsilon_l\omega_l$$

is still nondegenerate. If the ε_i are rational, then ω' defines a foliation $\mathcal{F}'$ (with $T\mathcal{F}' = \ker(\omega')$) for some fibration of M over S^1.

Claim. For sufficiently small ε_i (and ignoring details about boundaries), we have $e(T\mathcal{F}') = e(T\mathcal{F})$.

To prove the claim, choose an inner product on TM (i.e. a Riemannian metric on M). Then when the ε_i are sufficiently small, we have a projection $T_p\mathcal{F} \to T_p\mathcal{F}'$ that is an isomorphism at each $p \in M$ and varies continuously over M. Hence a section of $T\mathcal{F}$ projects to a section of $T\mathcal{F}'$ with the same zero set, and the claim follows.

To finish the proof, observe that if α' is the rational homology class dual to ω', we have

$$x(\alpha') = -\left\langle e(T\mathcal{F}'), \alpha'\right\rangle = -\langle e(T\mathcal{F}), \alpha\rangle,$$

as desired. $\square$