THREE-MANIFOLDS NOTES

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LAST LECTURE

Given $\alpha = [F, \partial F] \in H_2(M, \partial M; \mathbb{Z})$ fibered we have $F \to M^3 \to S^1$ and a foliation $\mathfrak{F}$ of M. There is an Euler class $e = e(T\mathfrak{F}) \in H^2(M, \partial M; \mathbb{Z})$. For $\beta \in H_2(M, \partial M; \mathbb{R})$ near to α then

$$X(\beta) = -\langle e, \beta \rangle. \tag{1}$$

OPEN FIBERED CONES

Today we show that every rational class in the open cone of a face of B_X containing the class $\frac{\alpha}{X(\alpha)}$ is fibered.

The closed cone is determined by equation (1)

$$\overline{C} = \{\beta \in H_2(M, \partial M; \mathbb{R}) | \, X(\beta) = -\langle e, \beta \rangle\}. \tag{2}$$

Proposition 1. *For any* $\beta \in \overline{C} \cap H_2(M, \partial M; \mathbb{Q})$ *and* $s, t \in \mathbb{Q}$ $s > 0$, *and* $t \geq 0$ *the class* $s\alpha + t\beta$ *is fibered.*

For simplicity we assume that M is closed. It suffices to consider $\beta \in \overline{C} \cap H_2(M, \partial M; \mathbb{Z})$ and let S be an essential norm-minimizing surface with $[S] = \beta$.

Lemma 2. S *is either isotopic to a leaf of* $\mathfrak{F}$ *or* S *is isotopic to a surface which is transverse to* $\mathfrak{F}$ *except for finitely many saddles.*

Proof. (Sketch) Uses ideas from foliation theory. Let $p : M \to S^1$ be the bundle map. We can assume that p_S is Morse by isotopy of S. Then $\mathfrak{F}$ induces a singular foliation of S. Each leaf is compact with at most one singularity (max, min, or saddle) and there are only finitely many singular leaves.

Starting with a max or a min, consider leaves expanding from the singularity until they meet a singular leaf. There are three possibilities

Date: 27 Apr. 2016.

(1) The leaves meet at another max or min. In this case there is a sphere component, which contradicts that S was essential. (With boundary the leaves could meet the boundary for a disk component.)

(2) The expanding leaves meet a saddle leaf and the closure of the non-singular part is a disk. In this case an isotopy cancels the max/min and the saddle singularity.

(3) The expanding leaves meet a singular leaf and touch the singular point twice. Consider slightly exterior (non-singular) leaves, which bound an annulus. Handle this case last. An isotopy makes the annulus flat, but p_S is no longer Morse.

Now we have to handle a few more cases. Expanding a boundary leaf of the annulus, if it meets a saddle leaf then we can isotopy to a flat pair of pants. There are a few more cases and manipulations. □

An alternative proof by Calegari uses minimal surfaces. The foliation is isotoped so every leaf is minimal, and S is isotoped to be minimal. Then it is shown that any max or min violates minimality.

Recall that our goal is to show that $X(\beta) = -\langle e, \beta \rangle$ implies that $s\alpha + t\beta$ is fibered when $s > 0$ and $t \geq 0$. If S is transverse then it is fibered so we only consider the case of S being transverse except at saddle points.

Let I_p be the number of positively oriented saddle points (the orientation of S agrees with the orientation of the leaf at the saddle), and I_n the number of negatively oriented saddle points.

Lemma 3.

(1) $\langle e, [S] \rangle = I_n - I_p$.

(2) $X_-(S) = I_p + I_n$.

Proof. (Idea) ii. is shown by the Poincaré-Hopf formula for the vector field corresponding to gradient of $p|_S$. For i. $e([S])$ may be considered as evaluating a vector field tangential to the induced foliation on S. Only the singular (saddle) leaves will contribute to the pairing and they evaluate as 1 if negatively oriented and -1 if positively oriented. □

Now if $X(\beta) = -\langle e, \beta \rangle$ then $I_n = 0$ because the left side is $I_p + I_n$ and the right side is $I_p - I_n$. Thus all tangencies are orientated consistent with the foliation. Let $\mathcal{X}$ be a vector field normal to S and transverse to $\mathfrak{F}$ in a neighborhood of S. Represent α by a 1-form ω which satisfies $\omega(\mathcal{X}) > 0$ point-wise in a neighborhood of S.

Let η be a closed 1-form, representing the Poincaré dual of S, supported in a neighborhood of S and satisfying $\eta(\mathcal{X}) > 0$ whenever η is non-zero. Then $s\omega + t\eta$ is Poincaré dual to $s\alpha + t\beta$ and if we show that it is non-degenerate that implies that $s\alpha + t\beta$ is fibered by Tichler's theorem.

In the support of η we have that $(s\omega + t\eta)(\mathcal{X}) > 0$ so it is non-degenerate. Outside of the support of η, ω is non-degenerate so this proves Proposition 1.

Corollary 4. *If M is irreducible, atoroidal, orientable, and $\beta_1 \geq 2$ (first Betti number), then M contains an essential surface which is not fibered.*

The surface corresponds to any vertex of B_X.

Summary

We have described $H_2(M, \partial M; \mathbb{R}) \cong H^1(M; \mathbb{R})$. The rational points of a disjoint union of cones corresponding to open faces of ∂B_X (Thurston cones) give all the fibered classes. These satisfy

(1) $\alpha, \beta \in C$ implies $\alpha + \beta \in C$,

(2) $X(\alpha + \beta) = X(\alpha) + X(\beta)$.