3-MANIFOLDS NOTES

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MAY 2, 2016

1. Hierarchies

Definition. M is Haken if it is orientable, irreducible, and contains a 2-sided π_1-injective surface $(F, \partial F) \subseteq (M, \partial M)$ which is not S^2.

Remark. This definition may disagree with the earlier one.

Exercise: If M is irreducible, oriented with $\partial M \neq \emptyset$, then M is Haken. Also, if $H^1(M) \neq 0$, then M is Haken.

Definition. A partial hierarchy for M is a sequence (finite or infinite)

$$\mathcal{H} : (M_1, F_1), \ldots, (M_n, F_n), \ldots$$

so that $M_1 = M$, $(F_i, \partial F_i) \subseteq (M_i, \partial M_i)$ where F_i is a 2-sided, π_1-injective, non boundary parallel, connected surface with regular neighborhood N_i and $M_i = M_{i-1} \backslash \dot{N}_{i-1}$.

$\mathcal{H}$ is a hierarchy if M_n is a union of 3-balls.

Example: Figure 1 denotes the hierarchy of the handlebody.

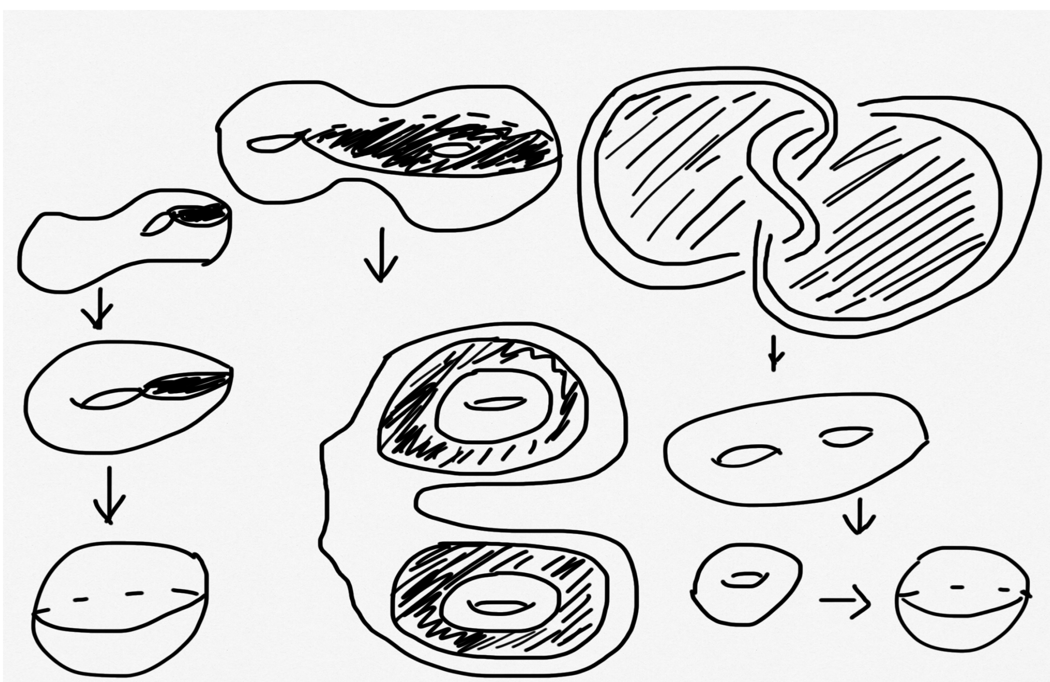

Figure 1

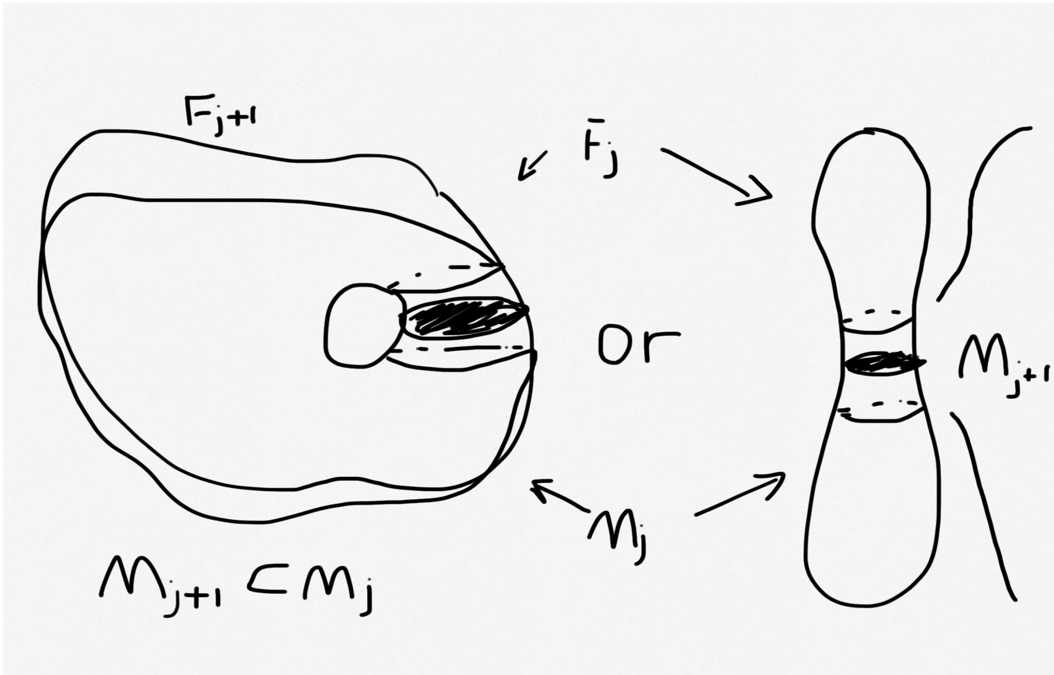

FIGURE 2

Lemma 1.1. *Let M be a compact 3-manifold. Let $\mathcal{H}$ be a partial hierarchy for M. If for all but finitely many F_i are disks, then $\mathcal{H}$ has finite length.*

Proof. Euler characteristic or complexity of boundary goes down when you cut along a disk. □

Lemma 1.2. *Let M be a compact 3-manifold. Let $\mathcal{H}$ be a partial hierarchy for M. If S is an incompressible, closed surface in M_i for some i, then S is also incompressible in M.*

Lemma 1.3. *Suppose $\mathcal{H}$ is a partial hierarchy for M with k surfaces and for some k, at least k of the surfaces F_i are not disks. Then there is a hierarchy $\mathcal{H}'$ where the first k surfaces are non-disks.*

Proof. Suppose $F_j = D^2$ whereas $F_{j+1} = D^2$. By a proper isotopy of $F_{j+1} \subset M_{j+1}$, we can ensure $\partial F_{j+1} \cap N_j = \varnothing$. So we may reorder two terms of the hierarchy

$$\ldots, (M_j, F_j), (M_{j+1}, F_{j+1}), \ldots$$

by

$$\ldots, (M_j, F_{j+1}), (M_{j+1}, F_j), \ldots$$

□

Recall the closed Haken number $\bar{h}(M)$ for an oriented irreducible and compact M is defined as follows,

Definition. $\bar{h}(M)$ is the maximal number of non-parallel closed incompressible 2-sided surfaces whose union embeds in M.

Theorem 1.4. *Suppose $\mathcal{H}$ is a partial hierarchy of an oriented irreducible and compact M. If for each i, the surface F_i is essential and connected, then there exists at most $3\bar{h}(M)$ surfaces F_i such that F_i is not a disk.*

Definition. The maximal number of non-disks in such a partial hierarchy is $v(M)$, which is the length of M.

Exercise: Find a 3-manifold with $v(M) = 3\bar{h}(M) \neq 0$

Definition. $\mathcal{D} = \sqcup D_i \subset M$ is a complete system of disks if the D_i are non-parallel, essential, and boundary components of $M \backslash \dot{N}(\mathcal{D})$ are π_1-injective.

Lemma 1.5. *Let $F \subset M$ be a non-disk essential surface. Then there is a complete system of disks disjoint from F.*

Now we prove Theorem 1.4 by contradiction.

Proof. By Lemma 1.3, there is a hierarchy

$$\ldots, (M_1, F_1), \ldots, (M_{3\bar{h}(M)+1}, F_{3\bar{h}(M)+1}), \ldots$$

in which all of these F_i's are non-disks. For each F_i, $i \leqslant 3\bar{h}(M)+1$, associate a closed incompressible surface as follows,

1. If F_i is closed, then set $S_i = F_i$.
2. If $F_i \subset M_i$ has non-empty boundary, by Lemma 1.4, let $\mathcal{D}_i$ be a complete system of disks for M_i. Let M_i' denote the component of $M_i \backslash \dot{N}(\mathcal{D}_i)$ containing F_i. Choose a component of $\partial M_i'$ meeting ∂F_i. Let S_i be a copy of that component, pushed slightly into $\dot{M}_i'$. Since S_i is an incompressible, closed surface in M_i, then S_i is incompressible in M.

At least four S_i's are parallel, which we call S_p, S_q, S_r, S_s. There are two cases here.

1. Suppose $S_p = F_p$ and $S_q = F_q$. Suppose $p < q$. Then S_q is ∂-parallel in M_q, a contradiction.
2. Suppose S_p, S_q, S_r are all associated to surface with boundary. Say S_p is between S_q and S_r. Suppose S_q lies on the F_p-side of S_p. If $q > p$, then $S_q \cap F_p = \varnothing$, so F_p is ∂-parallel, a contradiction. If $q < p$, symmetric reasoning.

□

Corollary 1.6. *If M is Haken, then M has hierarchy.*