3-MANIFOLDS

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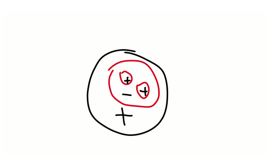
1. Sutured Manifolds

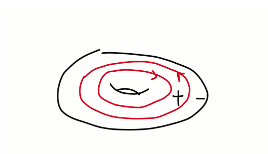
Definition. A **sutured manifold** is a 4-tuple (M, R_+, R_-, γ) (often abbreviated (M, γ)) where M is a 3-manifold, γ is a union of annuli in ∂M and $\partial M \backslash \mathring{\gamma} = R_1 \sqcup R_2$ where each component intersects both R_+ and R_-.

Example. 1)

2)



3)



4)

Definition. (M, γ) is **taut** if

1) M is irredicible
2) R_+ and R_- are incompressible and norm minimizing in $H_2(M, \gamma)$

In the above examples, 1 and 3 are taut while 2 and 4 are not taut.
Terminology Change: $(S, \partial S) \subseteq (M, \partial M)$ is compressible if

1) S is a 2-sphere bounding a ball, or
2) S admits a compressing disk

Otherwise, $(S, \partial S)$ is incompressible.

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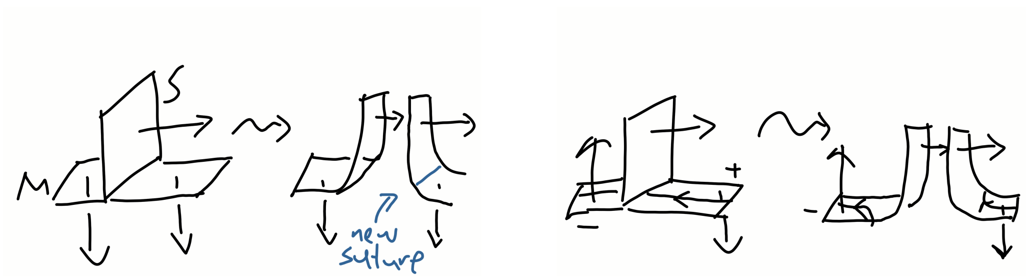
Remark. $[R_+] = -[R_-]$ in $H_2(M, \gamma)$ when taut.

Lemma 1.1. *$\gamma \subseteq \partial(D^2 \times S^1)$ a disjoint union of regular neighborhoods of oriented curves $\gamma_1, \ldots, \gamma_m$ determines a taut sutured manifold if and only if all of the following hold*

1) $n > 0$ is even
2) γ_i all (unoriented) parallel
3) γ_i doesn't bound a disk in $D^2 \times S^1$
4) Orientations "match up"

Lemma 1.2. *If $F \neq S^2$ is a compact, orientable surface then $(F \times I, F \times \{1\}, F \times \{0\}, \partial F \times I)$ is taut.*

Definition. A **sutured manifold decomposition** $(M, \gamma) \xrightarrow{S'} (M', \gamma')$ consists of a sutured manifold (M, γ), a properly embedded surface $(S, \partial S) \subseteq (M, \partial M)$ so $\partial S \cap \gamma$ is a union of arcs running between R_+ and R_- and so that (M', γ') is obtained as follows. $M' = M \backslash N_S$ where N_S is an open regular neighborhood of S. γ', R^+, R^- are obtained by choosing a co-orientation on $\partial M'$ in an "obvious" way. If it isn't obvious, then you are on a suture.



A **partial sutured manifold hierarchy** is a sequence

$$(M_1, \gamma_1) \xrightarrow{S_1} (M_2, \gamma_2) \xrightarrow{S_2} \ldots$$

and is a **sutured manifold hierarchy** if $M_n = \sqcup B^3$ for some n. The (partial) hierarchy is **taut** if each $(M_j, \gamma_j) \xrightarrow{S} (M_{j+1}, \gamma_{j+1})$ is **taut** in the sense that both (M_j, γ_j) and (M_{j+1}, γ_{j+1}) are taut.

Theorem 1.3 (Gabai, Scharlemann)**.** *If (M, γ) is taut, then it admits a taut sutured manifold hierarchy.*

Motivation: Use taut sutured manifold hierarchies to build taut foliations.
Idea: We obtain a foliation in which R_+ and R_- are unions of leaves and we would like to push the foliation up the hierarchy by gluing R_+ to R_- along S.

Theorem 1.4 (Thurston)**.** *A compact leaf of a taut foliation is norm minimizing.*

This was applied to compute genus of knots and links in S^3.

Recall. Length, $\nu(M)$, is the maximum number of non-disks in a hierarchy by essential surfaces.

Last time, we stated that $\nu(M) \leqslant 3\bar{h}(M)$ where $\bar{h}(M)$ is the maximum number of non-parallel closed essential surfaces which can be embedded at once.

Example. Let $\tau : T^2 \to T^2$ be defined by $\tau(z, w) = (-z, \bar{w})$. Let $M = T^2 \times I/(x, 1) \simeq (\tau(x), 1), (x, 0) \simeq (\tau(x), 0)$. We can check, using Seifert fibering, that $\bar{h}(M) = 1$. We get a hierarchy $(M, \text{horizontal torus}) \to E_1 \sqcup E_2$ where E_1 and E_2 are interval bundles over klein bottles. $\nu(M) = 3\bar{h}(M)$.