THREE-MANIFOLDS NOTES

DREW ZEMKE, EDITED BY BRIAN SHIN

PRIME DECOMPOSITION

Connect Sums.

Definition 1. If A and B are connected, oriented n-manifolds, then the *connect sum* of A and B, denoted $A\#B$, is obtained by choosing points $a \in A$ and $b \in B$, removing open balls around each point, and gluing the resulting boundary components by an orientation-reversing map.

The connect sum of two oriented manifolds is well-defined, which is to say that it does not depend on the choice of points, open balls, or the map between boundary components. In the smooth category, with some care we can extend the definition above to get a smooth connect sum of two smooth n-manifolds.

Exercise 1. If A and B are n-manifolds for $n \geq 2$ and R is any ring, show that

$$H_1(A\#B; R) \approx H_1(A; R) \oplus H_1(B; R).$$

Remark. If A and B are unoriented or nonorientable n-manifolds, it still makes sense to write "$M = A\#B$," as this expresses that there is an embedded $(n-1)$-sphere in M that cuts M into $A\backslash(\text{ball}) \cup B\backslash(\text{ball})$. However, since there may be multiple nonhomeomorphic M for which $M = A\#B$, it does not make sense to write "$A\#B$" by itself.

Definition 2. A 3-manifold M is *prime* if whenever $M = A\#B$, at least one of A and B is homeomorphic to S^3. A 3-manifold M is *irreducible* if every smoothly embedded 2-sphere in M bounds a 3-ball in M.

Notice that every irreducible 3-manifold is prime. It turns out that the converse is almost true:

Theorem 3. *Let M be an orientable 3-manifold. Then M is prime but not irreducible if and only if $M \cong S^1 \times S^2$.*

In the following, we use the notation $N(S)$ to denote a *regular neighborhood* of a submanifold S in a manifold M.

Proof. ($\Rightarrow$) If M is not irreducible then there is a 2-sphere $\Sigma \subset M$ that does not bound a ball. Since M is prime, Σ must be nonseparating. Hence there is an arc α that joins one boundary component of $N(\Sigma)$ the the other and does not meet $N(\Sigma)$ in its interior.

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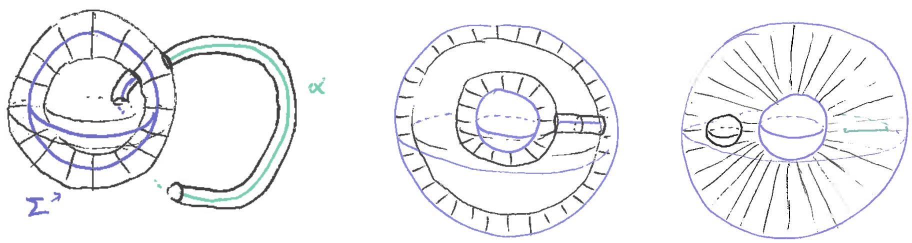
Let $N = N(\Sigma) \cup N(\alpha)$, and notice that ∂N is a 2-sphere. (It's the result of joining the two sphere boundary components of $N(\Sigma)$ by a tube.) In fact, N is homeomorphic to $S^1 \times S^2 \backslash$(3-ball). To see this, cut N open along Σ and expand the tube around alpha, as shown in the following figure.



Cut open along Σ (blue), then fill in all of the space inside the outer sphere except for a 3-ball.

Hence we have shown that M contains a 2-sphere ∂N that cuts M into $N \cong S^1 \times S^2 \backslash$ (3-ball) and another 3-manifold N' with a 2-sphere boundary component. Since M is prime, N' must be a 3-ball, and so we conclude that $M \cong S^1 \times S^2$.

($\Leftarrow$) Let $M = S^1 \times S^2$ and let $S = S^2 \times \{pt\} \subset M$. Then S is an embedded 2-sphere in M that does not bound a ball, so M is not irreducible. It remains to show that M is not prime.

Seeking a contradiction, suppose that $M = A \# B$ where neither A nor B is S^3. Then we can find a *reducing sphere* $T \subset M$, that is, an embedded separating 2-sphere in M for which neither component of $M \backslash T$ is a 3-ball. Isotope T in M to be transverse to S, so that $S \cap T$ is a smooth closed multicurve on S and T. We will assume that T has been chosen in its isotopy class so that the number of components of $S \cap T$ is minimized.

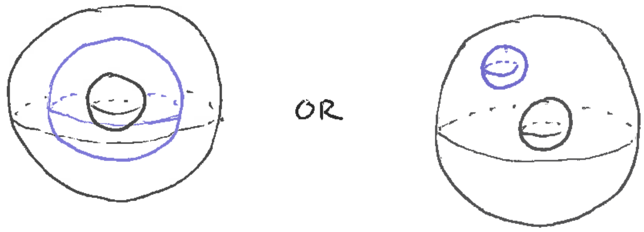
If $S \cap T \neq \emptyset$, choose a curve α of $S \cap T$ that is innermost on T. Then α bounds a disk $D \subset T$ such that $\mathrm{int}(D) \cap S = \emptyset$. Moreover, α cuts S into two disks D_1 and D_2.

Claim. One of $D \cup D_1$ and $D \cup D_2$ bounds a 3-ball in M.

> **Exercise 2.** Prove the claim using Alexander's theorem.

We can now use the ball from the claim to isotope T through S, which reduces the number of components of $S \cap T$ by at least one. This contradicts our assumption that the number of components was minimal, so we must have $S \cap T = \emptyset$.

It follows that T is a 2-sphere in $M \backslash S \cong S^2 \times I$. Furthermore, T is either isotopic to $S^2 \times \{pt\}$ — in which case it is nonseparating in M — or it bounds a 3-ball in $S^2 \times I$, and hence in M.



In either case, T is not a reducing sphere as claimed, so we have reached a contradiction. Thus M must be prime. $\square$

Remark. In the nonorientable case, there is also the possibility that M is the *twisted* S^2*-bundle over* S^1, denoted $S^1 \tilde{\times} S^2$.

Exercise 3. Extend the proof above to the nonorientable case.

The Prime Decomposition Theorem.

Theorem 4 (Kneser-Milnor). *If M is a compact and orientable 3-manifold, then M admits a* prime decomposition

$$M = P_1 \# \cdots \# P_k$$

where P_i is a prime 3-manifold for each i. Moreover, this decomposition is unique up to reordering.

Remark. There is a similar statement for nonorientable 3-manifolds. Some added care must be taken to deal with ambiguities arising from the fact that $A\#(S^1 \times S^2)$ is homeomorphic to $A\#(S^1 \tilde{\times} S^2)$ for any nonorientable 3-manifold A.

Before we start the proof, we note that every smooth n-manifold M admits a *smooth triangulation*, wherein every simplex $\tau \subset M$ has a chart in the smooth atlas that sends τ to the standard n-simplex in $\mathbb{R}^n$. It follows from this that if M is a compact smooth manifold, then rank $H_1(M;\mathbb{Z})$ and dim $H_1(M, \mathbb{Z}/2)$ are both finite.

Proof of Existence. Since $S^1 \times S^2$ contributes a $\mathbb{Z}$-summand to $H_1(M;\mathbb{Z})$ and $\mathbb{R}P^3$ contributes a $\mathbb{Z}/2$-summand to $H_1(M;\mathbb{Z}/2)$, by the preceding observation there can only be finitely many $S^1 \times S^2$ and $\mathbb{R}P^3$ connect summands. Hence we may assume that M has no such summands. We will also assume that none of the boundary components of M are 2-spheres, for if this were the case, we could eliminate such boundary components by splitting off 3-ball connect summands.

Choose a (smooth) triangulation of M with v vertices and f 2-simplices. We will show that if $M = M_1 \# \cdots \# M_N$ is any connect sum decomposition of M where $M_i \not\cong S^3$ for any i, then $N \leq v + f$. It will follow that M admits a prime decomposition.

Let $\Sigma \subset M$ be a system of $(N-1)$ 2-spheres in M that realizes the decomposition $M = M_1 \# \cdots \# M_N$. Thus the M_i are obtained from M by cutting M along Σ and capping off the resulting 2-sphere boundary components with 3-balls. We will call such a system of 2-spheres *independent* if no component of $M \setminus \Sigma$ is a punctured 3-sphere. (Hence the goal of this argument is bound the number of components of an independent system Σ.)

After an isotopy, we may assume that Σ is transverse to the triangulation of M. This means:

(i) Σ does not meet the vertices of the triangulation,

(ii) Σ meets the edges of the triangulation in isolated points on the interiors of the edges,

(iii) Σ meets the 2-simplices of the triangulation in disjoint curves and properly embedded arcs in the 2-cells.

We define the *complexity* of Σ to be $c(\Sigma) = (\alpha, \beta)$, where is α is the number of points of intersection of Σ with the 1-skeleton of the triangulation, and β is the number of components of intersection of Σ with each 2-cell, summed across all of the 2-cells of the triangulation. (Note: this is different from the number of components of the intersection of Σ with the 2-skeleton!) We order complexities lexicographically.

Suppose Σ has k components $S_1, \ldots, S_k$, and has minimal complexity among independent systems of 2-spheres.

To be continued!

$\square$