THREE-MANIFOLDS NOTES

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EXISTENCE OF PRIME DECOMPOSITION (CONT.)

Definition 1. (Non standard) Let M^3 be a smoothly triangulated 3-manifold with $\Sigma \subset M^3$ a smoothly, properly embedded, possibly disconnected, surface. Σ is called *semi-normal* if

(1) Σ is transverse to the triangulation (misses vertices, intersects edges and faces transversly).

(2) If σ is a 2-simplex of the triangulation, every component of $\Sigma \cap \sigma$ is an arc connecting distinct edges (i.e. not Figure 1a or Figure 1b.)

(3) If τ is a 3-simplex, every component of $\Sigma \cap \tau$ is a disc as in figure 2a. (No bounded genus and no pants as in Figure 2b.)

Aside: Σ is *normal* if it's semi-normal and for every 3-simplex τ, every component of $\Sigma \cap \partial\tau$ has combinatorial length (the number of faces it intersects in $\partial\tau$) less than or equal to 4.

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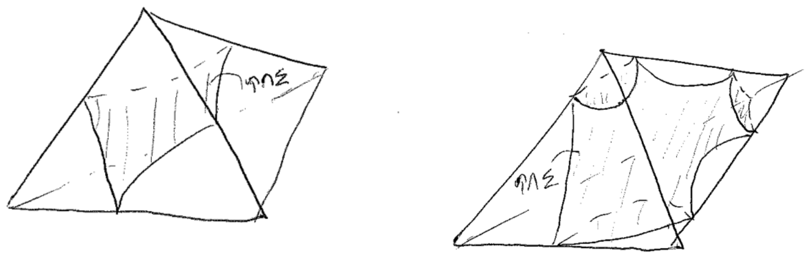
(A) 2-simplices should not contain loops.

(B) This also shouldn't happen.

FIGURE 1. Non-semi-normal 2-simplex pictures.



(A) Σ should intersect 3-simplices in discs.

(B) 3-simplices should not contain pants.

FIGURE 2. Good and bad 3-simplex pictures.

Most of the lecture will be devoted to proving the following proposition, which is at the heart of the existence of a prime decomposition.

Proposition 2. *If* M *is compact triangulated 3-manifold, and* $\Sigma \subset M$ *is an independent system of 2-spheres, with* $\#\Sigma = k$*, then there is an independent, semi-normal system of 2-spheres* $\Sigma' \subset M$ *with the same number of components.*

Recall that Σ is *independent* if no component of $M \backslash \Sigma$ is a punctured S^3.

Proposition 2 implies the existence of a prime decomposition for an orientable compact 3-manifold as follows (see also Lecture 3).

Proof of Existence. Suppose M is compact, orientable, and triangulated. Let $M = P_1 \# \cdots \# P_N$. Every $S^1 \times S^2$ summand of M contributes a $\mathbb{Z}$ direct summand to $H_1(M;\mathbb{Z})$, and every $\mathbb{R}P^3$ summand contributes a $\mathbb{Z}/2$ summand to $H_1(M;\mathbb{Z}/2)$. Since M is compact, it has finite first Betti number and therefore finitely many $S^1 \times S^2$ and $\mathbb{R}P^3$ summands. We split these summands off of M and henceforth assum that no connect summand of M is $S^1 \times S^2$ or $\mathbb{R}P^3$. Similarly, M has finitely many S^2 boundary components, and we may eliminate these by splitting off a B^3 connect summand for each S^2 boundary component.

From the connect sum decompostion $M = P_1 \# \cdots \# P_N$, we get an independent system Σ consisting of $(N-1)$ S^2's, from the connect sum decomposition. Proposition 2 implies there is a system with as many components that is semi-normal. Since each sphere is separating and no factor is $S^2 \times I$ or an I-bundle over $\mathbb{R}P^2$, each component of $M \backslash \Sigma$ intersects some 2-simplex in a non-rectangle, i.e. a piece different from the one shown in Figure 3. If each intersection was a rectangle than it would be an I-bundle over S^2 or $\mathbb{R}P^2$, which would correspond

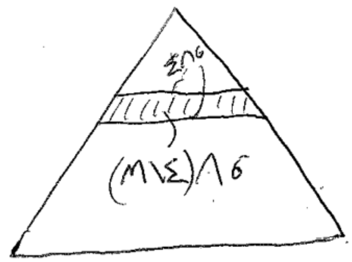


FIGURE 3. Intersection of a 2-simplex with a component of $M\backslash\Sigma$ that is a rectangle. .

to either a component of $M\backslash S$ that was $S^2 \times I$ (a twice-punctured S^3) or an $\mathbb{R}P^3$ summand of M.

Hence we have $N-1 \leq$ #(2-simplices) + #(vertices), and so we have shown that the number of connect summands in any decomposition of M is bounded. $\square$

The following lemma will help with the proof of Proposition 2.

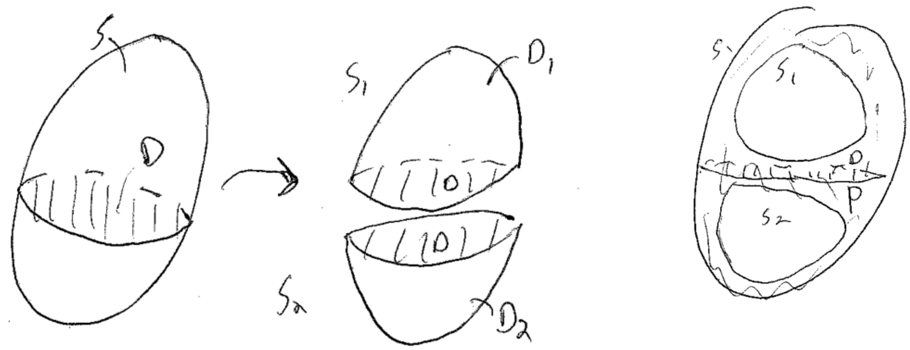
Lemma 3. *Let Σ be an independent system of separating 2-spheres in M, and let D be an embedded 2-disc with $D \cap \Sigma = \partial D$ that meets Σ in a 2-sphere S. Suppose D cuts S into discs D_1 and D_2, and let $S_1 = D_1 \cup D$ and $S_2 = D_2 \cup D$, as in Figure 4a. If $\Sigma_1 = (\Sigma\backslash D_2) \cup D$ and $\Sigma_2 = (\Sigma\backslash D_1) \cup D$, then either Σ_1 or Σ_2 is independent.*

Proof. Let P be a regular neighborhood of $S \cup D$ as in the lemma. Then $P \cong S^3\backslash(3\ B^3\text{'s})$ as in Figure 4b.

For $i = 1, 2$, let B_i be the components of $M\backslash(\Sigma \cup S_1 \cup S_2)$ meeting S_i. Because Σ is independent at least one of B_1 or B_2 is not a punctured S^3 (Exercise: if $N = N_1$ glued to N_2 along a disc, then N is a punctured S^3 if and only if N^1 and N^2 are punctured S^3's).

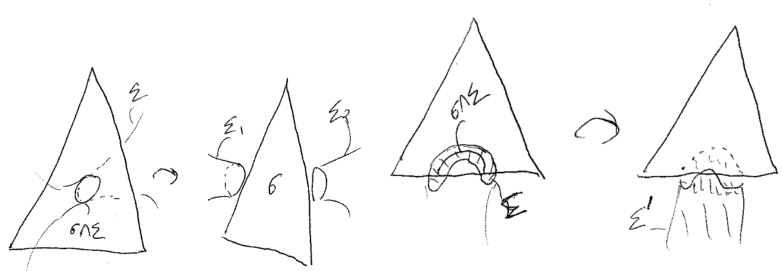
Suppose that B_1 is not a punctured S^3. Then Σ_1 is independent. Let A be the component of $M\backslash(\Sigma \cup S_1 \cup S_2)$ meeting S, so that A is not a punctured S^3 by independence of Σ. The only new pieces of the complement are $M\backslash\Sigma_1$ are B_1 (which is not a punctured S^3) and A glued along a disc to B_2. By the exercise, this new piece is not a punctured S^3. $\square$

Proof of Proposition 2. Define $c(\Sigma) = (\alpha, \beta)$ with $\alpha = \#(\Sigma \cap M^{(1)})$, and $\beta = \sum\{\#(\Sigma \cap \sigma)|\sigma \text{ is } 2-\text{simplex}\}$. Let $c(\Sigma) = \infty$ if Σ is not transverse to the triangulation of M. Order these pairs lexicographically.



(A) Surgery of S. (B) P is a punctured S^3.

FIGURE 4. Lemma 3 pictures.



(A) Surgery of a loop. (B) Surgery of an arc.

FIGURE 5. 2-simplex surgery pictures.

Claim: A smallest complexity independent system of a given cardinality is semi-normal.

Let Σ be such a system. It's transverse, $c(\mathbb{S}) < \infty$.

(1) $\Sigma \cap \sigma$ has no loops: Choose an innermost loop and surger along the disc that it bounds. By Lemma 3 we get a new independent system with $c(\Sigma') = (\alpha, \beta - 1)$. See Figure 5a.

(2) $\Sigma \cap \sigma$ contains no arcs: Choose an outermost arc and consider a regular neighborhood in Σ. Use a taco shape to form the other side of the baseball. This decreases α by 2 (and can be done by isotopy). See Figure 5b.

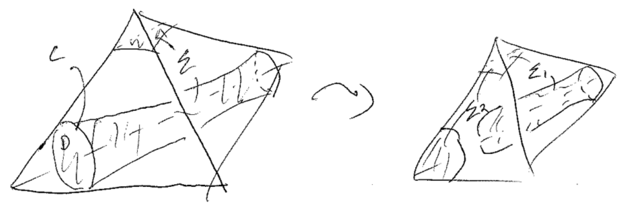


FIGURE 6. Surgery when $\Sigma \cap \tau$ is not a disc.

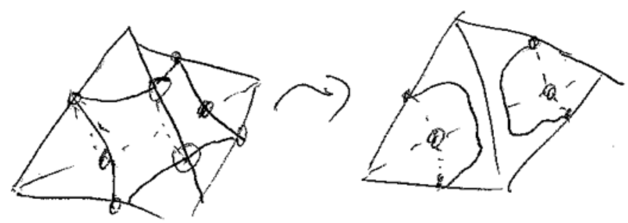


FIGURE 7. Surgery when Σ is semi-normal but not normal.

(3) Suppose $\Sigma \cap \tau$ has a (planar) non-disc component: Choose c innermost on $\partial\tau$ amongst curves of $\Sigma \cap \partial\tau$ bounding non-discs in τ. Then this bounds a disc in τ so surger with $D \cap \Sigma = c$ (pushing slightly inside of τ). By the lemma there is a new independent system and whichever it is, α decreases. See Figure 6.

This completes the proof of the proposition and the proof of the existence of a prime decomposition for orientable compact 3-manifolds. The non-orientable case is not too much more complicated. □

Remark. In fact the lowest complexity system is normal. Use the taco shape to move an extra edge outside and decrease α, figure 7.