THREE MANIFOLDS NOTES

JASON MANNING, NOTES BY ZHEXIU TU

1. Uniqueness of Prime Decomposition

Last time we showed the existence of prime decomposition. This class we are going to show the uniqueness of the prime decomposition.

Theorem 1.1. *Let M^3 be compact, connected, and orientable. Then there is a unique prime decomposition, up to insertion or deletion of S^3's, i.e., if P_i and Q_i are irreducible and not S^3 and*

$$\begin{aligned} M &\cong P_1\#\cdots\#P_n\#(\#_{i=1}^k S^1\times S^2) \\ &\cong Q_1\#\cdots\#Q_m\#(\#_{i=1}^l S^1\times S^2), \end{aligned}$$

then the list $\underline{Q}=(Q_1,\ldots,Q_m)$ is a permutation of $\underline{P}=(P_1,\ldots,P_n)$ and $k=l$.

Remark 1.2. For any non-orientable manifold, there is also a unique prime decomposition if you prohibit the $S^1\times S^2$ summands.

Proof. The existence of such a prime decomposition was shown in last class.

Definition 1.3. Let Σ be a system of 2-spheres in M. Σ *decomposes* M into $\underline{P}$ if the components of $M\backslash\Sigma$ which are not punctured 3-spheres are in bijective correspondence with the entries of $\underline{P}$, where the P_i's are punctured.

Suppose

$$\begin{aligned} M &\cong P_1\#\cdots\#P_n\#(\#_{i=1}^k S^1\times S^2) \\ &\cong Q_1\#\cdots\#Q_m\#(\#_{i=1}^l S^1\times S^2). \end{aligned}$$

We may assume that P_i and Q_i are not B^3 by factoring off any S^2 components of the ∂M. Then there are systems Σ_P,Σ_Q so that Σ_P decomposes M into $\underline{P}$, and Σ_Q decomposes M into $\underline{Q}$. Suppose $\#(\Sigma_P\cap\Sigma_Q)$ is minimal among pairs of such systems. (We may always suppose this is a transverse intersection.). We claim that

$$\#(\Sigma_P\cap\Sigma_Q)=0.$$

Suppose not. We choose a loop of intersection that is innermost on a sphere of Σ_Q, bounding a disk α. See Figure 1. Let $S\subset\Sigma_P$ be the sphere containing α. Let $\Sigma_P'=(\Sigma_P\backslash S)\cup(S_1\cup S_2)$. Note that $\#(\Sigma_P'\cap\Sigma_Q)<\#(\Sigma_P\cap\Sigma_Q)$ and Σ_P' still decomposes M into P by irreducibility of the components of $M\backslash\Sigma_P$.

So $\Sigma_P\cup\Sigma_Q$ decomposes M into $\underline{P}$ and decomposes M into $\underline{Q}$ (again using irreducibility of the components of $M\backslash\Sigma$). But the list of irreducible non-S^3 summands is recoverable from such a system (plug in balls to $\partial(M\backslash\Sigma)$ and throw away S^3's). So $\underline{P}=\underline{Q}$ up to reordering. To see $k=l$, note that

$$H_1(M)=H_1(\#\underline{P})\oplus\mathbb{Z}^k=H_1(\#\underline{Q})\oplus\mathbb{Z}^l.$$

Therefore $k=l$. □

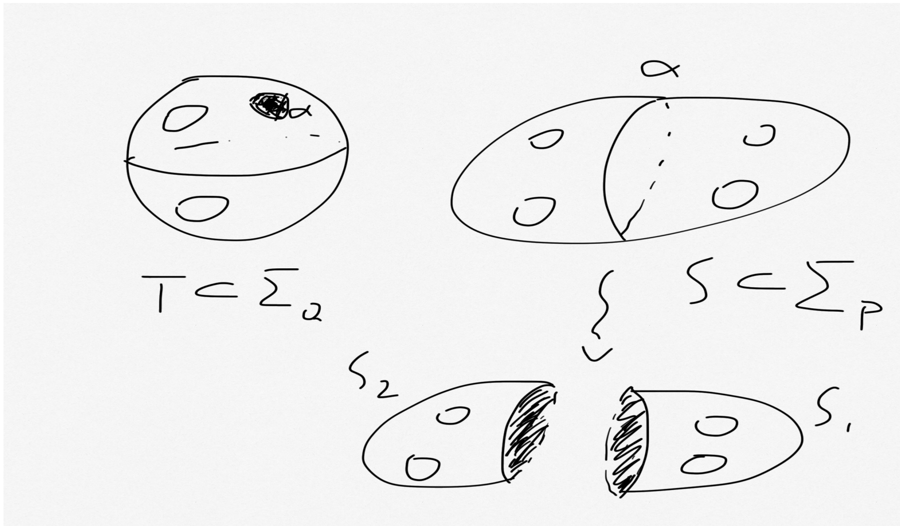

FIGURE 1

Proposition 1.4. *Suppose $p : \tilde{M} \to M$ is a covering of 3 manifolds and $\tilde{M}$ is irreducible. Then M is irreducible.*

Proof. We can show $p|_B : B \to p(B)$ is a covering map. The covering space is one sheeted on S^2, hence one sheeted on all of B. □

Remark 1.5. The converse is also true, but it requires "tower argument".

Corollary 1.6. *Any manifold M covered by S^3 on $\mathbb{R}^3$ is irreducible.*

Examples include $\mathbb{R}P^3$, lens spaces, Poincaré dodecahedral space, $\Sigma \times S^1$, where $\chi \leq 0$. Note that there exist irreducible, simply connected 3-manifolds $\not\cong S^3$ or $\mathbb{R}^3$. But they don't cover compact 3-manifolds (follows from geometrization.)

Theorem 1.7 (Waldhausen)**.** *The universal covering of a Haken 3-manifold is $\mathbb{R}^3$.*

Exercise: what's the universal cover of $L_1 \# L_2$ where L_1, L_2 are lens spaces?
Extensive reading: Whitehead manifold

2. Torus decomposition

Definition 2.1. $\Sigma^2 \subset M^3$ which is not a disk or a two sphere S^2 is called incompressible if every simple loop in Σ bounding a disk in M also bounds one in Σ.

If $\Sigma \neq S^2, D^2$ is not incompressible, we call it compressible. A compressible surface can be compressed. Figure 2 illustrates the innermost loop argument: there is an embedded disk whose boundary is essential in Σ, and we can get a simpler surface Σ' which is homologous to Σ.

Definition 2.2. Let M^3 be closed, irreducible, and orientable. We say M Haken if it contains an incompressible 2-sided surface.

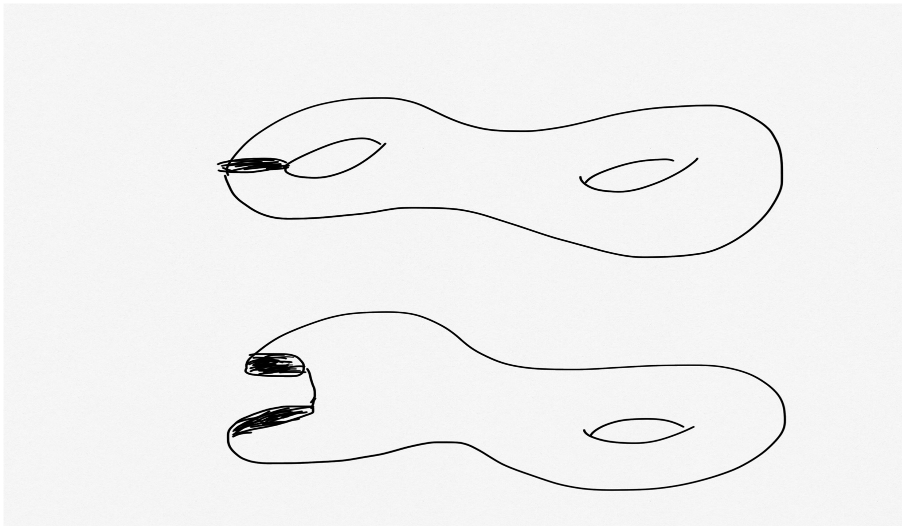

FIGURE 2

Remark 2.3. (1) $\chi(\Sigma) \leq 0$, $\Sigma \hookrightarrow M$ π_1-injective, then Σ is incompressible. (Converse requires Σ 2-sided & Loop Theorem)

(2) $\chi(\Sigma) \leq 0$, $\Sigma \hookrightarrow \mathbb{R}^3$ is always compressible. (Think about the proof of Alexander's Theorem)